



Yang–Baxter maps, discrete integrable equations and quantum groups

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Abstract

For every quantized Lie algebra there exists a map from the tensor square of the algebra to itself, which by construction satisfies the set-theoretic Yang–Baxter equation. This map allows one to define an integrable discrete quantum evolution system on quadrilateral lattices, where local degrees of freedom (dynamical variables) take values in a tensor power of the quantized Lie algebra. The corresponding equations of motion admit the zero curvature representation. The commuting Integrals of Motion are defined in the standard way via the Quantum Inverse Problem Method, utilizing Baxter’s famous commuting transfer matrix approach. All elements of the above construction have a meaningful quasi-classical limit. As a result one obtains an integrable discrete Hamiltonian evolution system, where the local equation of motion are determined by a classical Yang–Baxter map and the action functional is determined by the quasi-classical asymptotics of the universal **R**-matrix of the underlying quantum algebra. In this paper we present detailed considerations of the above scheme on the example of the algebra $U_q(sl(2))$ leading to discrete Liouville equations, however the approach is rather general and can be applied to any quantized Lie algebra.

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1. Introduction

The “Yang–Baxter maps” [1] are invertible maps of a Cartesian product of two identical sets \mathcal{X} ,

$$\mathcal{R}: \quad \mathcal{X} \times \mathcal{X} \mapsto \mathcal{X} \times \mathcal{X} \quad (1.1)$$

satisfying the “functional” or “set-theoretic” Yang–Baxter equation [2],

$$\mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23} = \mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12}. \quad (1.2)$$

This equation states an equality of two different composition of the three maps \mathcal{R}_{12} , \mathcal{R}_{13} and \mathcal{R}_{23} , acting on different factors in a product of three sets $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ (for instance, \mathcal{R}_{12} coincides with the map (1.1) in the first and second factors and acts as an identity map in the third one; the maps \mathcal{R}_{13} and \mathcal{R}_{23} are defined similarly).

The interest to the Yang–Baxter maps is motivated mostly by their connection to discrete integrable evolution equations. Important examples of the Yang–Baxter maps as well as some classification results were obtained in [3–11]. Mention also a related *consistency around a cube* condition [12], which is in many cases can be associated with some set-theoretic Yang–Baxter equation. One common limitation of the existing methods is that Hamiltonian structures of the arising maps generally remain unclear. As a result there are no regular procedures for quantization of the Yang–Baxter maps or their applications to Hamiltonian evolution systems.

In this paper we address these problems and present a new approach to the Yang–Baxter maps, which is based on the theory of quantum groups [13,14] and naturally connected to Baxter’s commuting transfer matrices [15] and the Quantum Inverse Problem Method [16]. The key role in our approach will be played by the universal \mathbf{R} -matrix [13]. This notion is associated with the so-called, *quasi-triangular Hopf algebras*, which include a very important class of the q -deformed (affine) Lie algebras and super-algebras (more precisely, their universal enveloping algebras). Let \mathcal{A} be a quasi-triangular Hopf algebra, then there exists an invertible element $\mathbf{R} \in \mathcal{A} \otimes \mathcal{A}$, belonging to the tensor product of two algebras \mathcal{A} , called the universal \mathbf{R} -matrix, which by construction satisfies the quantum Yang–Baxter equation

$$\mathbf{R}_{12} \mathbf{R}_{13} \mathbf{R}_{23} = \mathbf{R}_{23} \mathbf{R}_{13} \mathbf{R}_{12}, \quad \mathbf{R} \in \mathcal{A} \otimes \mathcal{A}. \quad (1.3)$$

Similarly to Eq. (1.2) the indices here indicate how the universal \mathbf{R} -matrix is embedded into the triple product $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ (for example, $\mathbf{R}_{12} \in \mathcal{A} \otimes \mathcal{A} \otimes 1$, and similarly for \mathbf{R}_{13} and \mathbf{R}_{23}). Interestingly, the universal \mathbf{R} -matrix allows one to construct a map from the tensor square of the algebra \mathcal{A} to itself [17,18],

$$\mathcal{R}: \quad \mathcal{A} \otimes \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}, \quad (1.4)$$

acting as

$$\mathcal{R}: \quad (x \otimes y) \mapsto (x' \otimes y') = \mathbf{R}(x \otimes y) \mathbf{R}^{-1}, \quad x, y \in \mathcal{A}. \quad (1.5)$$

It will automatically satisfy Eq. (1.2) in virtue of (1.3) and commutation relations of the quantum algebra \mathcal{A} . Thus, for any quasi-triangular algebra \mathcal{A} the map (1.5) is a solution the set-theoretic Yang–Baxter equation (1.2), where the algebra \mathcal{A} serves as a set \mathcal{X} . At this point it should be noted that a quasi-triangular algebra \mathcal{A} is, of course, not just an abstract structureless set (as it is implicitly assumed in the setting of Eq. (1.2)), but has algebraic relations between its elements, which must be taken into account in order for the Eq. (1.2) to hold. Obviously, this

is a slight generalization of the meaning of Eq. (1.2), but this is a well needed generalization, since the most important applications of the Yang–Baxter maps to dynamical systems naturally require to equip the set \mathcal{X} with an additional structure for purposes of quantization or considerations of Hamiltonian systems. Indeed, we show that the map (1.5) defines Hamiltonian evolution equations for a discrete integrable quantum system in 2D (with discrete space and time) with an algebra of observables formed by a tensor power of the quantum algebra \mathcal{A} . The functional Yang–Baxter equation (1.2) then implies that these evolution equations obey the quantum version of the consistency-around-a-cube condition [12]. Further, the map (1.5) possesses a discrete analog of the zero curvature representation. This allows one to construct a full set of mutually commuting integrals of motion using the standard approach of commuting transfer matrices [15] and the Quantum Inverse Problem Method [16]. The unitary Heisenberg evolution operator for the system are expressed through the matrix elements of the universal \mathbf{R} -matrix.

Remarkably, all steps of our new approach to the Yang–Baxter maps admit a meaningful quasi-classical limit. As a result for any quasi-triangular algebra \mathcal{A} one obtains a classical Yang–Baxter map which automatically possesses properties of a Hamiltonian map, since it preserves the tensor product structure of Poisson algebras arising in the quasi-classical limit of the quantum algebra \mathcal{A} . The action functional of the corresponding classical discrete integrable system is determined by the quasi-classical asymptotics of the universal \mathbf{R} -matrix.

In this paper we illustrate the above ideas for the case of the algebra $U_q(sl(2))$, which is the simplest example of a quasi-triangular Hopf algebra. The quantum case is considered in Sect. 2 and the classical one in Sect. 3. The considerations are completely parallel. For the reader's convenience we preserve the same numeration of the corresponding subsections in both cases. Note the discrete integrable equations arising here for the algebra $U_q(sl(2))$ are related to the discrete Liouville equation, various variants of which were previously considered in [19–22].

2. Quantum Yang–Baxter map for the algebra $U_q(sl(2))$

2.1. Algebra $U_q(sl(2))$

In this subsection we briefly summarize the basic properties of the algebra $U_q(sl(2))$. A more detailed description of this algebra well suited to our purposes can be found in [23]. The universal enveloping algebra $\mathcal{A} = U_q(sl(2))$ is generated by elements H, E, F , satisfying the relations,

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = (q - q^{-1})(q^H - q^{-H}). \quad (2.1)$$

Here we use a normalization, where the elements E and F are multiplied by the factor $(q - q^{-1})$ with respect to the usual choice. Below, it will be convenient to also use the element $K = q^H$, for which the commutation relations (2.2) become

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad [E, F] = (q - q^{-1})(K - K^{-1}). \quad (2.2)$$

The quadratic Casimir operator, which commutes with all other elements of the algebra, has the form

$$C = q^{-1}K + qK^{-1} + EF. \quad (2.3)$$

The algebra (2.2) is a Hopf algebra with the co-multiplication Δ , the co-unit ϵ , and the antipode S . The co-multiplication is a map from the algebra \mathcal{A} to its tensor square

$$\Delta: \quad \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}, \quad (2.4)$$

defined as

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F. \quad (2.5)$$

The co-unit ϵ is a map from \mathcal{A} to complex numbers. It is defined as

$$\epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0, \quad (2.6)$$

The co-multiplication and co-unit define algebra homomorphisms, i.e., $\Delta(ab) = \Delta(a) \Delta(b)$ and $\epsilon(ab) = \epsilon(a)\epsilon(b)$. The antipode S is defined as¹

$$S(K) = K^{-1}, \quad S(E) = -E K^{-1}, \quad S(F) = -K F. \quad (2.7)$$

Correspondingly, for the element H one has

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \epsilon(H) = 0, \quad S(H) = -H, \quad K = q^H. \quad (2.8)$$

Note that the antipode is an algebra anti-homomorphism, i.e., $S(ab) = S(b)S(a)$. It satisfies an important properties

$$S(1) = 1, \quad \epsilon \circ S = \epsilon, \quad (S \otimes S) \circ \Delta = \sigma \circ \Delta \circ S = \Delta' \circ S \quad (2.9)$$

where Δ' is another co-multiplication obtained from Δ by interchanging factors in the tensor product,

$$\Delta' = \sigma \circ \Delta, \quad \sigma(x \otimes y) = (y \otimes x). \quad (2.10)$$

2.2. Universal \mathbf{R} -matrix

The algebra $U_q(sl(2))$ is a *quasi-triangular* Hopf algebra. This means that there exists an element $\mathbf{R} \in \mathcal{A} \otimes \mathcal{A}$, called the universal \mathbf{R} -matrix, which satisfies the properties

$$\begin{aligned} \Delta'(x) \mathbf{R} &= \mathbf{R} \Delta(x), \quad \forall x \in \mathcal{A}, \\ (\Delta \otimes 1) \mathbf{R} &= \mathbf{R}_{13} \mathbf{R}_{23}, \\ (1 \otimes \Delta) \mathbf{R} &= \mathbf{R}_{13} \mathbf{R}_{12}, \end{aligned} \quad (2.11)$$

where $\mathbf{R}_{12} = \mathbf{R} \otimes 1$, $\mathbf{R}_{23} = 1 \otimes \mathbf{R}$ and $\mathbf{R}_{13} = (\sigma \otimes 1) \mathbf{R}_{23}$. Together with (2.7) and (2.9) this definition implies the quantum Yang–Baxter equation

$$\mathbf{R}_{12} \mathbf{R}_{13} \mathbf{R}_{23} = \mathbf{R}_{23} \mathbf{R}_{13} \mathbf{R}_{12}, \quad (2.12)$$

and two simpler relations

$$\begin{aligned} (\epsilon \otimes 1) \mathbf{R} &= (1 \otimes \epsilon) \mathbf{R} = (1 \otimes 1), \\ (S \otimes 1) \mathbf{R} &= (1 \otimes S^{-1}) \mathbf{R} = \mathbf{R}^{-1}. \end{aligned} \quad (2.13)$$

¹ More generally, the defining property of the antipode reads [23],

$$m \circ (S \otimes 1) \circ \Delta(a) = m \circ (1 \otimes S^{-1}) \circ \Delta(a) = 0$$

where m is multiplication map $m \circ (a \otimes b) = ab$ for algebra \mathcal{A} . For $\mathcal{A} = U_q(sl(2))$ this implies (2.7).

Note, that the universal \mathbf{R} -matrix is not unique as an element of $\mathcal{A} \otimes \mathcal{A}$. However, if one assumes, that $\mathbf{R} \in \mathcal{B}_- \otimes \mathcal{B}_+$, where \mathcal{B}_+ and \mathcal{B}_- are Borel subalgebras of \mathcal{A} , generated by the elements (H, F) and (H, E) respectively, then the universal \mathbf{R} -matrix is uniquely defined by the following formal series [13]

$$\mathbf{R} = q^{\frac{H \otimes H}{2}} \prod_{k=0}^{\infty} \left(1 - q^{2k+1} E \otimes F \right). \quad (2.14)$$

2.3. Quantum Yang–Baxter map

Let $X = \{K, E, F\}$ denotes the set of generating elements of the algebra \mathcal{A} and $X_{1,2}$ denote these sets in the corresponding components of the tensor product $\mathcal{A} \otimes \mathcal{A}$,

$$X_1 = X \otimes 1, \quad X_2 = 1 \otimes X, \quad X = \{K, E, F\}. \quad (2.15)$$

Let us now explicitly calculate the map (1.5). Evidently, it is completely determined by its action on the generating elements of the tensor product,

$$\mathcal{R}: \quad (X_1, X_2) \mapsto (X'_1, X'_2), \quad X'_i = \mathbf{R} X_i \mathbf{R}^{-1}, \quad i = 1, 2. \quad (2.16)$$

Using (2.14) and (2.2) one reproduces the result of [18]

$$\begin{cases} K'_1 = K_1 (1 - q^{-1} K_1^{-1} E_1 F_2 K_2), \\ E'_1 = E_1 K_2, \\ F'_1 = F_1 K_2^{-1} + F_2 - K_1^{-2} F_2 (1 - q K_1^{-1} E_1 F_2 K_2)^{-1}, \end{cases} \quad (2.17a)$$

and

$$\begin{cases} K'_2 = (1 - q^{-1} K_1^{-1} E_1 F_2 K_2)^{-1} K_2, \\ E'_2 = K_1 E_2 + E_1 - E_1 K_2^2 (1 - q K_1^{-1} E_1 F_2 K_2)^{-1}, \\ F'_2 = K_1^{-1} F_2. \end{cases} \quad (2.17b)$$

Note, that the relatively complicated expressions for F'_1 and E'_2 above can be easily obtained by combining the remaining four equations in (2.17) with the formulae for the co-multiplications Δ and Δ' .

For further references present also the inverse map. Solving (2.17) with respect to the “un-primed” variables, one obtains

$$\begin{cases} K_1 = K'_1 (1 - q^{-1} E'_1 F'_2)^{-1}, \\ E_1 = E'_1 (K'_2)^{-1} (1 - q^{-1} E'_1 F'_2)^{-1}, \\ F_1 = (F'_1 + (K'_1)^{-1} F'_2) (1 - q^{-1} E'_1 F'_2) K'_2 - K'_1 F'_2 K'_2, \end{cases} \quad (2.18a)$$

and

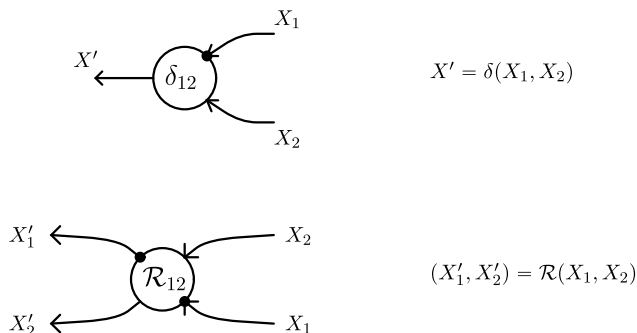


Fig. 1. Graphical representation of the maps (2.20) and (2.22). The first arguments are marked with heavy dots.

$$\begin{cases} K_2 = (1 - q^{-1} E'_1 F'_2) K'_2, \\ E_2 = (E'_2 + E'_1 K'_2)(1 - q^{-1} E'_1 F'_2) (K'_1)^{-1} - E'_1 (K'_1 K'_2)^{-1}, \\ F_2 = K'_1 F'_2 (1 - q^{-1} E'_1 F'_2)^{-1}. \end{cases} \quad (2.18b)$$

2.4. Properties of the quantum Yang–Baxter map

It is instructive to reformulate the properties of the universal **R**-matrix, stated above in (2.11)–(2.13), as properties of the quantum map (2.17).² For this purpose is it useful to introduce a “set-theoretic multiplication” δ which acts on two sets of generating elements

$$\delta : (X_1, X_2) \mapsto X', \quad (2.19)$$

and write it as

$$X' = \delta(X_1, X_2). \quad (2.20)$$

Explicitly, it is defined as

$$K' = K_1 K_2, \quad E' = E_1 K_2 + E_2, \quad F' = F_1 + K_1^{-1} F_2, \quad (2.21)$$

which are essentially the same formulae as in (2.5), but their meaning is rather different. Indeed, it is worth noting that the direction of the arrow in (2.19) is reversed with respect to (2.4) and that the map (2.19) is only defined on two sets of generating elements.

Below, it will also be convenient to write the Yang–Baxter map (2.16) in a functional form

$$(X'_1, X'_2) = \mathcal{R}(X_1, X_2). \quad (2.22)$$

Fig. 1 shows a graphical representation of the maps (2.20) and (2.22). Note that these map are not symmetric upon exchanging their arguments. To avoid an ambiguity their first arguments on the diagram are marked with heavy dots.

Consider the tensor product $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ of three algebras \mathcal{A} and let (X_1, X_2, X_3) denote three sets of generators in the corresponding components of the product. Define a functional operator \mathcal{R}_{12}

² As noted above this map was previously obtained in [18], however, its properties, given below in Eqs. (2.28)–(2.32) are new.

$$\mathcal{R}_{12}(X_1, X_2, X_3) := (X'_1, X'_2, X_3) = (\mathcal{R}(X_1, X_2), X_3) \quad (2.23a)$$

which acts as (2.16) on the first two sets and does not affect the third one. Similarly, define operators

$$\mathcal{R}_{13} = \sigma_{23} \circ \mathcal{R}_{12} \circ \sigma_{23}, \quad \mathcal{R}_{23} = \sigma_{12} \circ \mathcal{R}_{13} \circ \sigma_{12}. \quad (2.23b)$$

At this point it is worth commenting on a relationship between the operator notations used in Sect. 2.2, and the set-theoretic substitutions \mathcal{R}_{ij} defined here. Note, that successive similarity transformations acting on a list of generator sets $(X_1, X_2, X_3 \dots)$ with quantum operators \mathbf{R}_{ij} is equivalent to a successive application of the corresponding set-theoretic map \mathcal{R}_{ij} taken in the *reverse* order. For instance, one can easily verify

$$\mathbf{R}_{12} \mathbf{R}_{13} (X_1, X_2, X_3) \mathbf{R}_{13}^{-1} \mathbf{R}_{12}^{-1} \equiv \mathcal{R}_{13}(\mathcal{R}_{12}(X_1, X_2, X_3)), \quad (2.24)$$

$$\mathbf{R}_{12} \mathbf{R}_{13} \mathbf{R}_{23} (X_1, X_2, X_3) \mathbf{R}_{23}^{-1} \mathbf{R}_{13}^{-1} \mathbf{R}_{12}^{-1} \equiv \mathcal{R}_{23}(\mathcal{R}_{13}(\mathcal{R}_{12}(X_1, X_2, X_3))).$$

Eq. (2.12) then immediately implies that the set-theoretic maps \mathcal{R}_{ij} satisfy the functional Yang–Baxter equation

$$\mathcal{R}_{23}(\mathcal{R}_{13}(\mathcal{R}_{12}(X_1, X_2, X_3))) = \mathcal{R}_{12}(\mathcal{R}_{13}(\mathcal{R}_{23}(X_1, X_2, X_3))) \quad (2.25)$$

and, more generally,

$$\mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12} = \mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23}. \quad (2.26)$$

To reformulate the remaining properties of the universal \mathbf{R} -matrix introduce additional notations,

$$\delta_{12}(X_1, X_2, X_3) := (\delta(X_1, X_2), X_3), \quad \delta_{23}(X_1, X_2, X_3) := (X_1, \delta(X_2, X_3)). \quad (2.27)$$

Then Eqs. (2.11) lead to

$$\delta(X_2, X_1) = \delta \circ \mathcal{R}(X_1, X_2), \quad (2.28)$$

$$\mathcal{R}(\delta_{12}(X_1, X_2, X_3)) = \delta_{12}(\mathcal{R}_{23}(\mathcal{R}_{13}(X_1, X_2, X_3))), \quad (2.29)$$

$$\mathcal{R}(\delta_{23}(X_1, X_2, X_3)) = \delta_{23}(\mathcal{R}_{12}(\mathcal{R}_{13}(X_1, X_2, X_3))). \quad (2.30)$$

Further, from (2.13) it follows that

$$\mathcal{R} \circ (\epsilon \otimes 1) = (\epsilon \otimes 1),$$

$$\mathcal{R} \circ (1 \otimes \epsilon) = (1 \otimes \epsilon), \quad (2.31)$$

$$\mathcal{R} \circ (S \otimes S) = (S \otimes S) \circ \mathcal{R}^{-1}, \quad (2.32)$$

where the co-unit ϵ and the antipode S are defined in (2.7). We would like to stress that the above relations (2.28)–(2.32) are not specific to the algebra $U_q(sl(2))$ and must hold for any quasi-triangular Hopf algebra. In the case of $U_q(sl(2))$ they could be verified using the explicit form of the map \mathcal{R} and its inverse given in (2.17) and (2.18). The equations (2.28) and (2.30) are illustrated in Fig. 2.

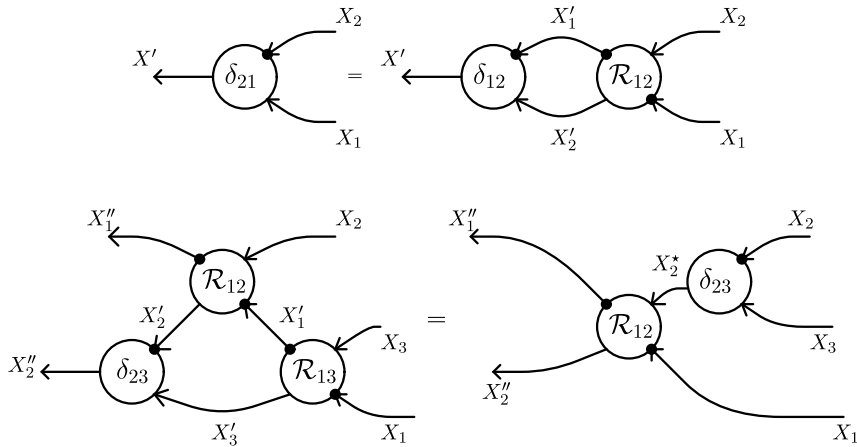


Fig. 2. Graphical representation of the relations (2.28) and (2.30).

2.5. \mathbf{R} -matrix form of the $U_q(sl(2))$ defining relations

As remarked before the universal \mathbf{R} -matrix is not uniquely defined by the relations (2.11). One can easily to check, that if $\mathbf{R}_{12} \in \mathcal{B}_- \otimes \mathcal{B}_+$, given by (2.14), satisfies (2.11), then so does the element

$$\mathbf{R}_{12}^* = \mathbf{R}_{21}^{-1} \in \mathcal{B}_+ \otimes \mathcal{B}_-, \quad (2.33)$$

which does not coincide with \mathbf{R}_{12} (this is why the algebra $U_q(sl(2))$ is called a *quasi-triangular* Hopf algebra). In particular, the new element \mathbf{R}^* satisfies the Yang–Baxter equation

$$\mathbf{R}_{12}^* \mathbf{R}_{13}^* \mathbf{R}_{23}^* = \mathbf{R}_{23}^* \mathbf{R}_{13}^* \mathbf{R}_{12}^*, \quad (2.34a)$$

which is a simple corollary of (2.12) and (2.33). Similarly, one can derive “mixed” relations

$$\mathbf{R}_{12}^* \mathbf{R}_{13} \mathbf{R}_{23} = \mathbf{R}_{23} \mathbf{R}_{13} \mathbf{R}_{12}^*, \quad (2.34b)$$

$$\mathbf{R}_{12} \mathbf{R}_{13} \mathbf{R}_{23}^* = \mathbf{R}_{23}^* \mathbf{R}_{13} \mathbf{R}_{12}, \quad (2.34c)$$

$$\mathbf{R}_{12} \mathbf{R}_{13}^* \mathbf{R}_{23} = \mathbf{R}_{23} \mathbf{R}_{13}^* \mathbf{R}_{12}, \quad (2.34d)$$

$$\mathbf{R}_{12}^* \mathbf{R}_{13}^* \mathbf{R}_{23} = \mathbf{R}_{23} \mathbf{R}_{13}^* \mathbf{R}_{12}^*. \quad (2.34e)$$

Let $\pi_{\frac{1}{2}}$ denotes the 2-dimensional (spin $\frac{1}{2}$) representations of the algebra (2.1),

$$\pi_{\frac{1}{2}}(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi_{\frac{1}{2}}(E) = (q - q^{-1}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi_{\frac{1}{2}}(F) = (q - q^{-1}) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.35)$$

Define the \mathbf{L} -operators

$$\mathbf{L}^+ = (\pi_{1/2} \otimes 1)(\mathbf{R}), \quad \mathbf{L}^- = (\pi_{1/2} \otimes 1)(\mathbf{R}^*), \quad (2.36)$$

by evaluating the universal \mathbf{R} -matrix in the 2-dimensional representation in the first space (to be called an auxiliary space). Using (2.14) and (2.33), one gets

$$\mathbf{L}^+ = \begin{pmatrix} q^{\frac{H}{2}} & q^{\frac{H}{2}} F \\ 0 & q^{-\frac{H}{2}} \end{pmatrix}, \quad \mathbf{L}^- = \begin{pmatrix} q^{-\frac{H}{2}} & 0 \\ -E q^{-\frac{H}{2}} & q^{\frac{H}{2}} \end{pmatrix}, \quad (2.37)$$

which are operator-valued matrices, whose elements belong to $U_q(sl(2))$ and act in the “quantum space”. Choosing the two-dimensional representation for the latter

$$R^+ = (1 \otimes \pi_{1/2})(\mathbf{L}^+) = (\pi_{1/2} \otimes \pi_{1/2})(\mathbf{R}), \quad R^- = (1 \otimes \pi_{1/2})(\mathbf{L}^-) = (\pi_{1/2} \otimes \pi_{1/2})(\mathbf{R}^*), \quad (2.38)$$

one gets the block R -matrices

$$R^+ = q^{-\frac{1}{2}} \begin{pmatrix} q & & & \\ & 1 & (q - q^{-1}) & \\ - & - & - & - \\ & & 1 & \\ & & & q \end{pmatrix}, \quad R^- = q^{+\frac{1}{2}} \begin{pmatrix} q^{-1} & & & \\ & 1 & & \\ - & - & - & - \\ & (q^{-1} - q) & 1 & \\ & & & q^{-1} \end{pmatrix}, \quad (2.39)$$

where the internal blocks act in the second space of the product $\mathbb{C}^2 \otimes \mathbb{C}^2$. It is convenient to define the parameter-dependent \mathbf{L} -operator

$$\mathbf{L}(\lambda) = \lambda \mathbf{L}^+ - \lambda^{-1} \mathbf{L}^-, \quad (2.40)$$

and the R -matrix

$$R(\lambda) = \lambda R^+ - \lambda^{-1} R^-. \quad (2.41)$$

The latter, of course, coincides with the R -matrix of the six-vertex model [24]. It possesses an important property

$$R(q^{\frac{1}{2}}) = (q - q^{-1}) P, \quad (2.42)$$

where P is the permutation matrix, interchanging the factors of the product $\mathbb{C}^2 \otimes \mathbb{C}^2$. It is useful to define

$$\check{R}^{\pm} = R^{\pm} P, \quad \check{R}(\lambda) = R(\lambda) P. \quad (2.43)$$

Let us now specialize the Yang–Baxter equations (2.12), and (2.34) by choosing the 2-dimensional representations (2.35) in the first and second spaces and then using (2.36) and (2.38). The matrix elements of the \mathbf{L} -operators will then act in the third space, which now becomes the quantum space. As a result one obtains the “ \mathbf{R} -matrix form” [25] of the commutation relations (2.1)

$$\begin{aligned} \check{R}^{\pm} (\mathbf{L}^+ \dot{\otimes} \mathbf{L}^+) &= (\mathbf{L}^+ \dot{\otimes} \mathbf{L}^+) \check{R}^{\pm}, & \check{R}^{\pm} (\mathbf{L}^- \dot{\otimes} \mathbf{L}^-) &= (\mathbf{L}^- \dot{\otimes} \mathbf{L}^-) \check{R}^{\pm}, \\ \check{R}^+ (\mathbf{L}^+ \dot{\otimes} \mathbf{L}^-) &= (\mathbf{L}^- \dot{\otimes} \mathbf{L}^+) \check{R}^+, & \check{R}^- (\mathbf{L}^- \dot{\otimes} \mathbf{L}^+) &= (\mathbf{L}^+ \dot{\otimes} \mathbf{L}^-) \check{R}^-_{12}. \end{aligned} \quad (2.44)$$

The symbol $\dot{\otimes}$ here denotes the tensor product of two-by-two matrices \mathbf{L}^{\pm} , acting in different factors of $\mathbb{C}^2 \otimes \mathbb{C}^2$, and operator product for their matrix elements. With an account of (2.42)

and the definitions (2.40), (2.41), the above relations can be combined into a compact form of the parameter-dependent Yang–Baxter equation

$$\check{R}(\lambda/\mu) (\mathbf{L}(\lambda) \dot{\otimes} \mathbf{L}(\mu)) = (\mathbf{L}(\mu) \dot{\otimes} \mathbf{L}(\lambda)) \check{R}(\lambda/\mu), \quad (2.45)$$

which is well-known in the theory of the six-vertex model (see [26,27] for recent advances in this area). Finally, note that the co-multiplication Δ , defined in (2.5), can be written as

$$\Delta(\mathbf{L}^\pm) = \mathbf{L}^\pm \otimes \mathbf{L}^\pm \quad (2.46)$$

where the symbol \otimes denotes the matrix product of the two-by-two matrices (2.37) and the tensor product for their operator-valued elements, belonging to different copies of the algebra $U_q(sl(2))$. There are two equations in (2.46), where the superscripts are either all pluses or all minuses, simultaneously.

2.6. Heisenberg–Weyl realization

The algebra $U_q(sl(2))$ admits an important homomorphism into the Heisenberg–Weyl algebra, \mathbf{W}_q , generated by two elements \mathbf{u} and \mathbf{v} ,

$$\mathbf{W}_q : \quad \mathbf{u} \mathbf{v} = q^2 \mathbf{v} \mathbf{u}. \quad (2.47)$$

Namely, if one sets

$$K = \mathbf{u}, \quad E = \mathbf{v}(z - q\mathbf{u}), \quad F = \mathbf{v}^{-1}(1 - qz^{-1}\mathbf{u}^{-1}), \quad (2.48)$$

where the element z commutes with \mathbf{u} and \mathbf{v} , then all commutation relations (2.2) will be satisfied in virtue of (2.47). The commuting element z parametrizes the Casimir operator (2.3),

$$C = z + z^{-1}. \quad (2.49)$$

Let us now consider the representation (2.48) as a change of variable for the set of generators $\{K, E, F\}$ to the set $\{\mathbf{u}, \mathbf{v}, z\}$ and rewrite the map (2.17) in the new variables. Assume the same meaning to the subscripts 1, 2 as in (2.17) (they distinguish components of the tensor product $\mathcal{A} \otimes \mathcal{A}$), for instance,

$$\mathbf{u}'_1 = \mathcal{R}(\mathbf{u}_1) = \mathbf{R}(\mathbf{u} \otimes 1) \mathbf{R}^{-1}, \quad \mathbf{u}'_2 = \mathcal{R}(\mathbf{u}_2) = \mathbf{R}(1 \otimes \mathbf{u}) \mathbf{R}^{-1}, \quad (2.50)$$

and similarly for $\mathbf{v}_{1,2}$ and $z_{1,2}$. From (2.17) it follows then

$$\begin{cases} \mathcal{R}(\mathbf{u}_1) = \mathbf{u}_1 \mathbf{g}, & \mathcal{R}(\mathbf{u}_2) = \mathbf{g}^{-1} \mathbf{u}_2, \\ \mathcal{R}(\mathbf{v}_1) = \left(\mathbf{v}_2^{-1} + (\mathbf{v}_1^{-1} - \frac{q}{z_2} \mathbf{v}_2^{-1}) \mathbf{u}_2^{-1} \right)^{-1}, & \mathcal{R}(\mathbf{v}_2) = \frac{z_1}{z_2} \mathbf{v}_1 + (\mathbf{v}_2 - \frac{q}{z_2} \mathbf{v}_1) \mathbf{u}_1, \\ \mathcal{R}(z_1) = z_1, & \mathcal{R}(z_2) = z_2, \end{cases} \quad (2.51)$$

where we have used the notation

$$\mathbf{g} = 1 - q^{-1} \mathbf{u}_1^{-1} \mathbf{v}_1 (z_1 - q \mathbf{u}_1) \mathbf{v}_2^{-1} (\mathbf{u}_2 - q z_2^{-1}). \quad (2.52)$$

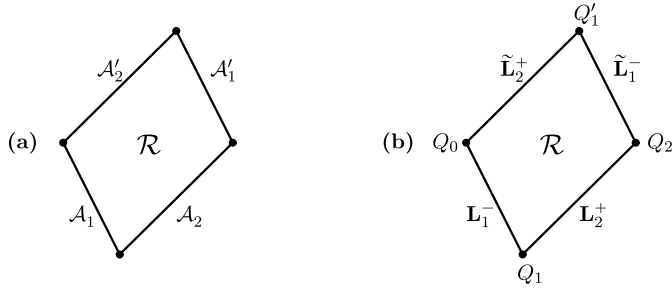


Fig. 3. (a) graphical illustration of the map \mathcal{R} , defined by (2.55), showing the assignment of the algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}'_1, \mathcal{A}'_2$ to the edges of a quadrilateral. (b) graphical illustration for the zero curvature relation (2.67b). The L -operators play a role of edge transition matrices for an auxiliary linear problem.

Note that the commuting elements z_1 and z_2 , obviously, remain invariant. They enter the map (2.51) as (spectral) parameters. Similarly for the inverse map, one obtains

$$\left\{ \begin{array}{ll} \mathcal{R}^{-1}(\mathbf{u}_1) = \mathbf{u}_1 \tilde{\mathbf{g}}^{-1}, & \mathcal{R}^{-1}(\mathbf{u}_2) = \tilde{\mathbf{g}} \mathbf{u}_2, \\ \mathcal{R}^{-1}(\mathbf{v}_1) = \left(\frac{z_1}{z_2} \mathbf{v}_2^{-1} + (\mathbf{v}_1^{-1} - \frac{z_1}{q} \mathbf{v}_2^{-1}) \mathbf{u}_2 \right)^{-1}, & \mathcal{R}^{-1}(\mathbf{v}_2) = \mathbf{v}_1 + (\mathbf{v}_2 - \frac{z_1}{q} \mathbf{v}_1) \mathbf{u}_1^{-1}, \\ \mathcal{R}^{-1}(z_1) = z_1, & \mathcal{R}^{-1}(z_2) = z_2, \end{array} \right. \quad (2.53)$$

where

$$\tilde{\mathbf{g}} = 1 - q^{-1} \mathbf{v}_1 (z_1 - q \mathbf{u}_1) \mathbf{v}_2^{-1} (1 - q z_2^{-1} \mathbf{u}_2^{-1}). \quad (2.54)$$

2.7. Discrete quantum evolution system

Here we introduce a $(1+1)$ -dimensional integrable quantum evolution system with discrete space and time. In doing this we follow the scheme suggested in [28,29] and further developed in [30–32]. Remind that the quantum Yang–Baxter map (2.16) acts on a tensor square of the algebra \mathcal{A} ,

$$\mathcal{R}: (\mathcal{A}_1 \otimes \mathcal{A}_2) \mapsto (\mathcal{A}'_1 \otimes \mathcal{A}'_2) = \mathbf{R}(\mathcal{A}_1 \otimes \mathcal{A}_2) \mathbf{R}^{-1}. \quad (2.55)$$

It can be represented graphically as in Fig. 3(a). We use this map to define a discrete quantum evolution system, where the algebra of observables

$$\mathcal{O} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{2N-1} \otimes \mathcal{A}_{2N}, \quad N \geq 1, \quad (2.56)$$

is formed by a tensor power of the algebra \mathcal{A} (all factors here are just independent copies of the algebra \mathcal{A} ; the indices indicate positions of the corresponding factors in the product). Let

$$\tilde{\mathcal{R}}_{ij} = \sigma_{ij} \circ \mathcal{R}_{ij} \quad (2.57)$$

denotes a composition of the map (2.55), acting on the i -th and j -th factors of the product (2.56), followed by the permutation operator σ_{ij} , swapping these factors with each other. With this notation we can define the map

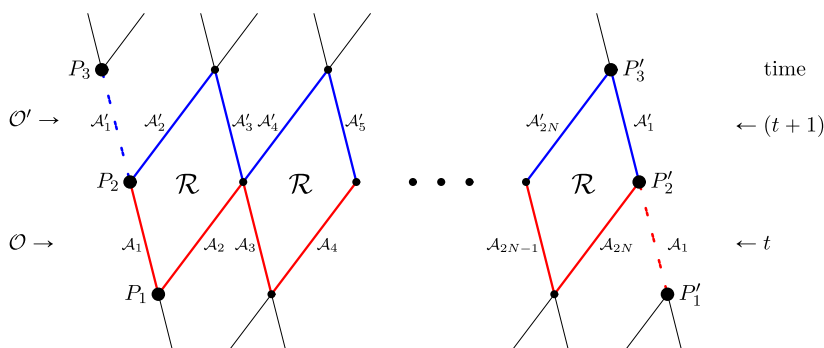


Fig. 4. Assignment of dynamical variables to edges of the square lattice. Due to periodic boundary conditions the points P_1, P_2, P_3 on the left side should be identified with their images P'_1, P'_2, P'_3 on the right side of the lattice.

$$\mathcal{U} = \mathcal{S} \circ (\tilde{\mathcal{R}}_{12} \circ \tilde{\mathcal{R}}_{34} \circ \cdots \circ \tilde{\mathcal{R}}_{(2N-1), 2N}), \quad (2.58)$$

where the operator \mathcal{S} cyclically shifts the factors in the product (2.56),

$$\mathcal{S}: \quad \mathcal{A}_n \rightarrow \mathcal{A}_{n+1}, \quad \mathcal{A}_{2N+1} \equiv \mathcal{A}_1. \quad (2.59)$$

The map (2.58) defines an evolution for one step of the discrete time. By construction it preserves the tensor product structure of the algebra of observables

$$\mathcal{O}' = \mathcal{U}(\mathcal{O}) = \mathcal{A}'_1 \otimes \mathcal{A}'_2 \otimes \cdots \otimes \mathcal{A}'_{2N-1} \otimes \mathcal{A}'_{2N}, \quad (2.60)$$

where each algebra \mathcal{A}_n is isomorphic to the algebra \mathcal{A} . So the evolution map is an automorphism of the algebra of observables, which preserves its tensor product structure (often referred to as an “ultra-local” structure). For a geometric interpretation consider the square lattice drawn diagonally as in Fig. 4, with $2N$ sites per row and periodic boundary conditions in the horizontal (spatial) direction. Now assign the factors of the product (2.56) to the set of connected edges forming a horizontal “saw”, as shown in Fig. 4. In the same way assign the algebras \mathcal{A}'_n in the product of (2.60) to edges of the saw, shifted by one time step above from the initial saw. Note that these conventions are completely consistent with the graphical representation of the map (2.55), corresponding to one quadrilateral, shown in Fig. 3(a).

The elements of the algebra (2.56) constitute the set of dynamical variables of the system at any fixed value of time. With the Heisenberg–Weyl realization (2.48) the generating elements of this algebra are expressed through the elements $\{\mathbf{u}_i, \mathbf{v}_i, z_i\}$, where the index $i = 1, 2, \dots, 2N$ is considered as the spatial coordinate, numerating the factors in (2.56). Combining (2.51) with (2.58) and (2.60), it is easy to see that

$$\mathcal{U}(z_{2n+1}) = z_{2n-1}, \quad \mathcal{U}(z_{2n}) = z_{2n}, \quad n = 1, 2, \dots, N, \quad (2.61)$$

where $z_{2N+1} \equiv z_1$. For simplicity, consider the homogeneous case, when

$$z_{2n-1} \equiv z_1, \quad z_{2n} \equiv z_2, \quad \forall n, \quad (2.62)$$

with arbitrary $z_{1,2}$. Then all z ’s stay unchanged under the time evolution, while the remaining variables transforms as

$$\left\{ \begin{array}{lcl} \mathcal{U}(\mathbf{u}_{2n+1}) & = & \mathbf{u}_{2n-1} \mathbf{g}_n, \\ \mathcal{U}(\mathbf{v}_{2n+1}) & = & \left(\mathbf{v}_{2n}^{-1} + (\mathbf{v}_{2n-1}^{-1} - \frac{q}{z_2} \mathbf{v}_{2n}^{-1}) \mathbf{u}_{2n}^{-1} \right)^{-1}, \\ \mathcal{U}(\mathbf{u}_{2n}) & = & \mathbf{g}_n^{-1} \mathbf{u}_{2n}, \\ \mathcal{U}(\mathbf{v}_{2n}) & = & \frac{z_1}{z_2} \mathbf{v}_{2n-1} + (\mathbf{v}_{2n} - \frac{q}{z_2} \mathbf{v}_{2n-1}) \mathbf{u}_{2n-1}, \end{array} \right. \quad (2.63)$$

where $n = 1, 2, \dots, N$,

$$\mathbf{g}_n = 1 - q^{-1} \mathbf{u}_{2n-1}^{-1} \mathbf{v}_{2n-1} (z_1 - q \mathbf{u}_{2n-1}) \mathbf{v}_{2n}^{-1} (\mathbf{u}_{2n} - q z_2^{-1}). \quad (2.64)$$

and the cyclic boundary conditions $\mathbf{u}_{2N+1} = \mathbf{u}_1$ and $\mathbf{v}_{2N+1} = \mathbf{v}_1$ are implied. With this realization the algebra of observables at any fixed moment of the discrete time reduces to the (spatially localized) tensor product of the Heisenberg–Weyl algebras (2.47), exactly as one would expect for the equal-time commutation relations of the canonical variables in a discrete quantum system.

2.8. Zero curvature representation

Consider again the Yang–Baxter map (2.55). Let \mathbf{L}_1^\pm and \mathbf{L}_2^\pm denote \mathbf{L} -operators (2.37), with elements belonging respectively to the first and second algebra of the tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$, appearing in (2.55). Next, recall the Yang–Baxter equation for the universal \mathbf{R} -matrix, given in (2.12). Cyclically shifting its indices $(1, 2, 3) \rightarrow (3, 1, 2)$ and then choosing the 2-dimensional representation (2.35) in the (new) space 3, one obtains

$$\mathbf{L}_1^+ \mathbf{L}_2^+ = \mathbf{R}_{12} \mathbf{L}_2^+ \mathbf{L}_1^+ \mathbf{R}_{12}^{-1}. \quad (2.65)$$

This equation involves a matrix product of \mathbf{L} 's (as two-by-two matrices) and a tensor product for their operator-valued matrix elements. Without the use of the indices 1, 2, the same equation can be written as

$$\mathbf{L}^+ \otimes \mathbf{L}^+ = \mathbf{R} (1 \otimes \mathbf{L}^+) (\mathbf{L}^+ \otimes 1) \mathbf{R}^{-1}, \quad (2.66)$$

however we will prefer the indexed form (2.65) for greater clarity. Further, let $\tilde{\mathbf{L}}_1^\pm$ and $\tilde{\mathbf{L}}_2^\pm$ are defined by the same equations (2.37), but with elements belonging, respectively, to the algebras \mathcal{A}'_1 and \mathcal{A}'_2 , appearing in (2.55). Then combining (2.65) and (2.55), one arrives to the equation

$$\mathbf{L}_1^+ \mathbf{L}_2^+ = \tilde{\mathbf{L}}_2^+ \tilde{\mathbf{L}}_1^+. \quad (2.67a)$$

Similarly, from (2.34b) and (2.34e) one obtains

$$\mathbf{L}_1^- \mathbf{L}_2^+ = \tilde{\mathbf{L}}_2^+ \tilde{\mathbf{L}}_1^-, \quad \mathbf{L}_1^- \mathbf{L}_2^- = \tilde{\mathbf{L}}_2^- \tilde{\mathbf{L}}_1^-. \quad (2.67b)$$

For instance, the first equation on the last line reads

$$\begin{aligned} (K_1 K_2)^{-\frac{1}{2}} \begin{pmatrix} K_2 & K_2 F_2 \\ -q E_1 K_2 & K_1 (1 - q^{-1} K_1^{-1} K_2 E_1 F_2) \end{pmatrix} &= \\ = (K'_1 K'_2)^{-\frac{1}{2}} \begin{pmatrix} (1 - q^{-1} E'_1 F'_2) K'_2 & K'_1 K'_2 F'_2 \\ -q E'_1 & K'_1 \end{pmatrix}. \end{aligned} \quad (2.68)$$

This equation, as well as the other two matrix equations in (2.67) are simple corollaries of the explicit expression of the Yang–Baxter map, given in (2.17). Conversely, the equations (2.67) can be used as an alternative definition of the Yang–Baxter map independent of the notion of the universal \mathbf{R} -matrix.

Evidently, Eqs. (2.67) provide a “Zero Curvature Representation” (ZCR) for the map (2.17) and for the local equations of motion of the discrete quantum evolution system considered in the previous subsection. Indeed, as illustrated in Fig. 3(b), the two-by-two matrices \mathbf{L} can be viewed as edge transition matrices for an auxiliary discrete linear problem. Eqs. (2.67) then imply that products of transition matrices along the paths $Q_0 Q_1 Q_2$ and $Q_0 Q'_1 Q_2$ are exactly the same, i.e., independent of the choice of path between Q_0 and Q_2 .

Using the notation (2.40) it is convenient to rewrite the ZCR (2.67) in the form

$$\mathbf{L}_1(\lambda) \mathbf{L}_2^+ = \tilde{\mathbf{L}}_2^+ \tilde{\mathbf{L}}_1(\lambda), \quad \mathbf{L}_1^- \mathbf{L}_2(\lambda) = \tilde{\mathbf{L}}_2(\lambda) \tilde{\mathbf{L}}_1^-, \quad (2.69)$$

where λ is arbitrary.

2.9. Commuting integrals of motion

The phase space of the quantum evolution system (2.60) possesses $4N$ degrees of freedom ($2N$ coordinates and $2N$ momenta). Let us show that the system has exactly $2N$ linearly independent mutually commuting integrals of motion (IM). Introduce two transfer matrices

$$\mathbf{T}(\lambda) = \text{Tr} \left(\mathbf{L}_1(\lambda) \mathbf{L}_2^+ \mathbf{L}_3(\lambda) \mathbf{L}_2^+ \cdots \mathbf{L}_{2N-1}(\lambda) \mathbf{L}_{2N}^+ \right) = \text{Tr} \prod_{n=1}^N \left(\mathbf{L}_{2n-1}(\lambda) \mathbf{L}_{2n}^+ \right) \quad (2.70a)$$

and

$$\bar{\mathbf{T}}(\lambda) = \text{Tr} \left(\mathbf{L}_1^- \mathbf{L}_2(\lambda) \mathbf{L}_3^- \mathbf{L}_4(\lambda) \cdots \mathbf{L}_{2N-1}^- \mathbf{L}_{2N}(\lambda) \right) = \text{Tr} \prod_{n=1}^N \left(\mathbf{L}_{2n-1}^- \mathbf{L}_{2n}^+(\lambda) \right). \quad (2.70b)$$

It follows from (2.44) and (2.45) that these transfer matrices form a commuting family

$$[\mathbf{T}(\lambda), \mathbf{T}(\lambda')] = [\bar{\mathbf{T}}(\lambda), \bar{\mathbf{T}}(\lambda')] = [\mathbf{T}(\lambda), \bar{\mathbf{T}}(\lambda')] = 0. \quad (2.71)$$

Further, using the zero curvature representation (2.69) one can easily show that the transfer matrices $\mathbf{T}(\lambda)$ and $\bar{\mathbf{T}}(\lambda)$ are, in fact, time-independent, i.e., they are unchanged under the discrete evolution map (2.58),

$$\mathbf{T}(\lambda) = \mathcal{U}(\mathbf{T}(\lambda)), \quad \bar{\mathbf{T}}(\lambda) = \mathcal{U}(\bar{\mathbf{T}}(\lambda)). \quad (2.72)$$

Let $(H_k, E_k, F_k) \in \mathcal{A}_k$ be the set of generators belonging to the k -th algebra in the tensor product (2.56). Define the operators

$$\begin{aligned} \mathbf{V}_k^+ &= q^{-1} E_k q^{-H_1-H_2-\dots-H_k}, & \bar{\mathbf{V}}_k^+ &= q^{-1} E_k q^{-H_k-H_{k+1}-\dots-H_{2N}}, \\ \mathbf{V}_k^- &= q^{-1} F_k q^{+H_1+H_2+\dots+H_k}, & \bar{\mathbf{V}}_k^- &= q^{-1} F_k q^{+H_k+H_{k+1}+\dots+H_{2N}}, \end{aligned} \quad (2.73)$$

which obey the following relations

$$\mathbf{V}_k^\sigma \mathbf{V}_\ell^{\sigma'} = q^{-2\sigma\sigma'} \mathbf{V}_\ell^{\sigma'} \mathbf{V}_k^\sigma, \quad \bar{\mathbf{V}}_k^\sigma \bar{\mathbf{V}}_\ell^{\sigma'} = q^{+2\sigma\sigma'} \bar{\mathbf{V}}_\ell^{\sigma'} \bar{\mathbf{V}}_k^\sigma, \quad k > \ell, \quad (2.74)$$

$$[\mathbf{V}_k^\sigma, \bar{\mathbf{V}}_k^\sigma] = 0, \quad [\mathbf{V}_k^\sigma, \bar{\mathbf{V}}_\ell^{\sigma'}] = 0, \quad k \neq \ell, \quad (2.75)$$

where $\sigma, \sigma' = \pm 1$ and

$$[\mathbf{V}_k^+, \bar{\mathbf{V}}_k^-] = [\bar{\mathbf{V}}_k^+, \mathbf{V}_k^-] = (q - q^{-1})(q^{H_k} - q^{-H_k}). \quad (2.76)$$

To within simple factors the transfer matrices (2.70) are N -th degree polynomials in the variable λ^2 (or λ^{-2}),

$$\mathbf{T}(\lambda) = \lambda^N \sum_{n=0}^N \lambda^{-2n} \mathbf{G}_n, \quad \bar{\mathbf{T}}(\lambda) = \lambda^{-N} \sum_{n=0}^N \lambda^{2n} \bar{\mathbf{G}}_n \quad (2.77)$$

whose coefficients are commuting integrals of motion.

$$[\mathbf{G}_k, \mathbf{G}_m] = [\bar{\mathbf{G}}_k, \mathbf{G}_m] = [\bar{\mathbf{G}}_k, \bar{\mathbf{G}}_m]. \quad (2.78)$$

From the definition (2.70) it is easy to see that there are only $2N$ independent coefficients, since two pairs of them coincide

$$\mathbf{G}_0 = \bar{\mathbf{G}}_0 = q^{\mathbf{P}} + q^{-\mathbf{P}}, \quad \mathbf{G}_N = \bar{\mathbf{G}}_N, \quad \mathbf{P} = \frac{1}{2} \sum_{k=1}^{2N} H_k. \quad (2.79)$$

The remaining coefficients can be expressed as “ordered sums” of products of the operators (2.73), which are analogous to the ordered integrals that appear in the context of the continuous conformal field theory [33] (the so-called non-local integrals of motion). For instance, the simplest non-trivial coefficients read

$$\mathbf{G}_1 = - \sum_{n=1}^N \left\{ q^{\mathbf{P}-H_{2n-1}} + q^{-\mathbf{P}+H_{2n-1}} - q^{\mathbf{P}+1} \sum_{\ell=1}^{2n-2} \mathbf{V}_\ell^- \mathbf{V}_{2n-1}^+ - q^{-\mathbf{P}+1} \sum_{\ell=2n}^{2N} \mathbf{V}_{2n-1}^+ \mathbf{V}_\ell^- \right\}, \quad (2.80)$$

$$\bar{\mathbf{G}}_1 = - \sum_{n=1}^N \left\{ q^{+\mathbf{P}-H_{2n}} + q^{-\mathbf{P}+H_{2n}} - q^{\mathbf{P}-1} \sum_{\ell=1}^{2n-1} \bar{\mathbf{V}}_\ell^+ \bar{\mathbf{V}}_{2n}^- - q^{-\mathbf{P}-1} \sum_{\ell=2n+1}^{2N} \bar{\mathbf{V}}_{2n}^- \bar{\mathbf{V}}_\ell^+ \right\} \quad (2.81)$$

Finally note that the transfer matrices (2.70) can be viewed as modified versions of the transfer matrix of the six-vertex model [24].

2.10. Matrix elements of the quantum \mathbf{R} -matrix

Here we present calculations of the matrix elements of the universal \mathbf{R} -matrix for the representations (2.48), when the element \mathbf{v} is diagonal. Various parts of these calculations previously appeared in [34–39]. The resulting expression (2.41) is remarkably simple and essentially coincide with that obtained in [37]. Define new parameters

$$q = e^{i\pi b^2}, \quad \eta = \frac{1}{2}(b + b^{-1}), \quad \text{Im}(b^2) > 0, \quad (2.82)$$

which are simply related to the value of $q < 1$. We will also assume $|b| < 1$. Introduce the Heisenberg algebra

$$[\mathbf{x}, \mathbf{p}] = \frac{i}{2\pi}. \quad (2.83)$$

Below, it will be convenient to represent the operators entering (2.48) in the form

$$\mathbf{u} = e^{-2\pi b \mathbf{p}}, \quad \mathbf{v} = e^{2\pi b (\mathbf{x} + i\beta)}, \quad z = -e^{2\pi i b (\alpha - \beta)}, \quad (2.84)$$

where α and β are arbitrary (complex) parameters. Note, that the parameter β can be eliminated by a trivial redefinition,

$$E \rightarrow e^{-2\pi i \beta} E, \quad F \rightarrow e^{+2\pi i \beta} F, \quad \alpha \rightarrow \alpha + \beta, \quad (2.85)$$

however, we prefer to retain it for further convenience. Next, consider the coordinate representations of the Heisenberg algebra (2.83) in the space of functions $f(s)$, quadratically integrable

$$\int_{ic-\infty}^{ic+\infty} |f(s)|^2 ds < \infty, \quad |\operatorname{Re}(c)| \leq |\operatorname{Re}(b)| \quad (2.86)$$

along any shifted real line $\operatorname{Re}(s) \in \mathbb{R}$ in the strip $|\operatorname{Im}(s)| \leq \operatorname{Re}(b)$, where the operators \mathbf{x} and \mathbf{p} acts as the multiplication and differentiation, respectively,

$$\mathbf{x} |s\rangle = s |s\rangle, \quad \mathbf{p} |s\rangle = -\frac{i}{2\pi} \frac{d}{ds} |s\rangle. \quad (2.87)$$

The corresponding representation of $U_q(sl(2))$, given by (2.48), (2.84) and (2.87), is irreducible [37] (see also [41]). It will be denoted as $\pi_{\alpha, \beta}$. It is spanned by the vectors $|s, \alpha, \beta\rangle$, where³

$$\begin{aligned} \mathbf{v} |s, \alpha, \beta\rangle &= e^{2\pi b (s + i\beta)} |s, \alpha, \beta\rangle, \quad \mathbf{u} |s, \alpha, \beta\rangle = |s - ib, \alpha, \beta\rangle, \\ z |s, \alpha, \beta\rangle &= -e^{2\pi i b (\alpha - \beta)} |s, \alpha, \beta\rangle. \end{aligned} \quad (2.88)$$

Further, let

$$|s_1, s_2\rangle = |s_1, \alpha_1, \beta_1\rangle \otimes |s_2, \alpha_2, \beta_2\rangle \quad (2.89)$$

denotes basic vectors in the tensor product $\pi_{\alpha_1, \beta_1} \otimes \pi_{\alpha_2, \beta_2}$ of two such representations, where for brevity the dependence on the parameters $\alpha_{1,2}$ and $\beta_{1,2}$ in the LHS of (2.89) is not explicitly shown. It is useful to introduce additional notations

$$v_i = e^{2\pi b (s_i + i\beta_i)}, \quad v'_i = e^{2\pi b (s'_i + i\beta_i)}, \quad z_i = -e^{2\pi i b (\alpha_i - \beta_i)} \quad i = 1, 2. \quad (2.90)$$

Equations (2.50) and (2.51) provide recurrence relations for the matrix elements of the \mathbf{R} -matrix. For instance, consider the second equation in the left column of (2.51), namely

$$\mathbf{R} \mathbf{v}_2 = \left(\frac{z_1}{z_2} \mathbf{v}_1 + \left(\mathbf{v}_2 - \frac{q}{z_2} \mathbf{v}_1 \right) \mathbf{u}_1 \right) \mathbf{R}. \quad (2.91)$$

If this equation is “sandwiched” in between the vectors $\langle s_1, s_2 |$ and $|s'_1, s'_2\rangle$ it gives

$$\langle s_1, s_2 | \mathbf{R} |s'_1, s'_2\rangle \left(v'_2 - \frac{z_1}{z_2} v_1 \right) = \left(v_2 - \frac{q}{z_2} v_1 \right) \langle s_1 + ib, s_2 | \mathbf{R} |s'_1, s'_2\rangle. \quad (2.92)$$

In this way four similar equations in (2.51) and (2.53) lead to the following four recurrence relations

³ Note, that after the redefinition (2.85) and some other simple equivalence transformations the representation $\pi_{\alpha, \beta}$ can be reduced to the representation π_s defined by Eqs. (2.4) of ref. [37] with $s = i\alpha$.

$$\begin{aligned}
\frac{\langle s_1 + i\mathbf{b}, s_2 | \mathbf{R} | s'_1, s'_2 \rangle}{\langle s_1, s_2 | \mathbf{R} | s'_1, s'_2 \rangle} &= \frac{z_1 \left(1 - \frac{z_2 v'_2}{z_1 v_1} \right)}{q \left(1 - \frac{z_2 v_2}{q v_1} \right)}, \\
\frac{\langle s_1, s_2 - i\mathbf{b} | \mathbf{R} | s'_1, s'_2 \rangle}{\langle s_1, s_2 | \mathbf{R} | s'_1, s'_2 \rangle} &= \frac{z_2 \left(1 - \frac{v_2}{v'_1} \right)}{q \left(1 - \frac{z_2 v_2}{q v_1} \right)}, \\
\frac{\langle s_1, s_2 | \mathbf{R} | s'_1 + i\mathbf{b}, s'_2 \rangle}{\langle s_1, s_2 | \mathbf{R} | s'_1, s'_2 \rangle} &= \frac{\left(1 - \frac{v_2}{v'_1} \right)}{q z_1 \left(1 - \frac{v'_2}{q z_1 v'_1} \right)}, \\
\frac{\langle s_1, s_2 | \mathbf{R} | s'_1, s'_2 - i\mathbf{b} \rangle}{\langle s_1, s_2 | \mathbf{R} | s'_1, s'_2 \rangle} &= \frac{\left(1 - \frac{z_2 v'_2}{z_1 v_1} \right)}{q z_2 \left(1 - \frac{v'_2}{q z_1 v'_1} \right)}. \tag{2.93}
\end{aligned}$$

We assume that the matrix elements $\langle s_1, s_2 | \mathbf{R}_{12} | s'_1, s'_2 \rangle$ are analytic in the strip $-\mathbf{b} \leq \text{Im} s_i, \text{Im} s'_i \leq \mathbf{b}$. Then the above recurrence relations uniquely define these matrix elements up to an inessential normalization factor,

$$\langle s_1, s_2 | \mathbf{R}_{12} | s'_1, s'_2 \rangle = V_{\beta_1 - \alpha_2}(s_2 - s_1) V_{\alpha_1 - \beta_2}(s'_2 - s'_1) \overline{V}_{\alpha_1 - \alpha_2}(s'_2 - s_1) \overline{V}_{\beta_1 - \beta_2}(s_2 - s'_1), \tag{2.94}$$

where

$$V_\alpha(s) = e^{i\pi/8 - i\pi s^2} \varphi(i\alpha - s), \quad \overline{V}_\alpha(s) = \frac{e^{-i\pi/8 + i\pi s^2}}{\varphi(i\alpha - i\eta - s)}. \tag{2.95}$$

The function $\varphi(s)$ is the “non-compact” quantum dilogarithm [36]

$$\varphi(s) = \exp \left(\frac{1}{4} \int_{\mathbb{R} + i0} \frac{e^{-2isw}}{\sinh(w\mathbf{b}) \sinh(w/\mathbf{b})} \frac{dw}{w} \right). \tag{2.96}$$

and the parameter η is defined in (2.82). Note that essentially the same expression (2.94) (but in a slightly modified form) was also obtained in ref. [37].

The non-compact quantum dilogarithm satisfies the quantum pentagon identity [22]

$$\varphi(\mathbf{p})\varphi(\mathbf{x}) = \varphi(\mathbf{x})\varphi(\mathbf{p} + \mathbf{x})\varphi(\mathbf{x}), \tag{2.97}$$

where \mathbf{x}, \mathbf{p} are the elements of Heisenberg algebra (2.83).

The functions $s V_\alpha(s)$ and $\overline{V}_\alpha(s)$ are neither positively defined nor symmetric with respect to the substitution $s \rightarrow -s$. However, they are Fourier-dual to each other

$$\int_{\mathbb{R}} e^{2\pi i s x} V_\alpha(s) ds = \overline{V}_\alpha(x). \tag{2.98}$$

The factorized form for the kernel of the \mathbf{R} -matrix corresponds to a factorized operator expression acting in the product of two Heisenberg–Weyl algebra (2.47) (but not the full algebra $U_q(sl(2))$),

$$\mathbf{R}_{12} = V_{\beta_1 - \alpha_2}(\mathbf{x}_2 - \mathbf{x}_1) V_{\alpha_1 - \alpha_2}(-\mathbf{p}_1) V_{\beta_1 - \beta_2}(\mathbf{p}_2) V_{\alpha_1 - \beta_2}(\mathbf{x}_1 - \mathbf{x}_2) \mathbf{P}_{12}, \quad (2.99)$$

where \mathbf{P}_{12} is the permutation operator,

$$\mathbf{P}_{12}|s_1, \alpha_1, \beta_1\rangle \otimes |s_2, \alpha_2, \beta_2\rangle = |s_2, \alpha_1, \beta_1\rangle \otimes |s_1, \alpha_2, \beta_2\rangle, \quad (2.100)$$

which swaps the coordinates s_1 and s_2 , but keep the parameters α_i and β_i unchanged.⁴ This \mathbf{R} -matrix coincides with that for the Dehn twist in quantum Teichmüller theory [39]. Explicit correspondence will be established in Appendix B. The same \mathbf{R} -matrix was also obtained in [40] in a similar framework. The elements of the matrix inverse to (2.94) are given by

$$\langle s'_1, s'_2 | \mathbf{R}^{-1} | s_1, s_2 \rangle = \frac{\bar{V}_{\alpha_1 - \alpha_2}^*(s'_2 - s_1) \bar{V}_{\beta_1 - \beta_2}^*(s_2 - s'_1)}{V_{\beta_1 - \alpha_2}(s_2 - s_1) V_{\alpha_1 - \beta_2}(s'_2 - s'_1)}, \quad (2.101)$$

where

$$\bar{V}_\alpha^*(s) = e^{i\pi/8 - i\pi s^2} \varphi(i\eta + i\alpha - s). \quad (2.102)$$

This function possesses the property

$$\lim_{\epsilon \rightarrow 0} \int ds \bar{V}_{\alpha + \epsilon}(s - u) \bar{V}_{\alpha - \epsilon}^*(s - v) = \delta(u - v), \quad (2.103)$$

where α is purely imaginary and a real positive ϵ defines circumventions of the poles. Note, that in the case when η is real and the spectral parameter α is purely imaginary, the superscript ‘*’ can be viewed as the complex conjugation. The relation (2.103) ensures the fact that the matrices (2.94) and (2.101) are indeed mutually inverse.

The \mathbf{R} -matrix (2.94) solves the quantum Yang–Baxter equation⁵:

$$\begin{aligned} & \int_{\mathbb{R}^3} ds'_1 ds'_2 ds'_3 \langle s_1, s_2 | \mathbf{R}_{12} | s'_1, s'_2 \rangle \langle s'_1, s_3 | \mathbf{R}_{13} | s'_1, s'_3 \rangle \langle s'_2, s'_3 | \mathbf{R}_{23} | s''_2, s''_3 \rangle \\ &= \int_{\mathbb{R}^3} ds'_1 ds'_2 ds'_3 \langle s_2, s_3 | \mathbf{R}_{23} | s'_2, s'_3 \rangle \langle s_1, s'_3 | \mathbf{R}_{13} | s'_1, s'_3 \rangle \langle s'_1, s'_2 | \mathbf{R}_{12} | s'_1, s'_2 \rangle. \end{aligned} \quad (2.104)$$

The statement follows from the Yang–Baxter equation (2.12) for the universal \mathbf{R} -matrix (2.14) and the irreducibility of the representation of $U_q(sl(2))$, given by (2.48) and (2.88). It is worth noting that the some solution (2.94) can be obtained as a limiting case of the \mathbf{R} -matrix of the Faddeev–Volkov model [38,42]. Thanks to this connection Eq. (2.104) can be independently verified using the various star–triangle relations, associated with the Faddeev–Volkov model and its reductions (see Appendix A for further details).

⁴ It should be note that the factorization of the type (2.94), (2.99) only happens for a specific choice of the basis of representations of $U_q(sl(2))$. It was first observed in [34] for the \mathbf{R} -matrix of the chiral Potts model, related to cyclic representations of $U_q(\widehat{sl}(2))$.

⁵ The \mathbf{R} -matrix (2.94) contains two sets of spectral parameters (α_1, β_1) and (α_2, β_2) , corresponding to the two representations appearing in (2.89), therefore we write it as $\langle s_1, s_2 | \mathbf{R}_{12} | s'_1, s'_2 \rangle$. In a similar manner the matrices $\langle s_1, s_3 | \mathbf{R}_{13} | s'_1, s'_3 \rangle$ and $\langle s_2, s_3 | \mathbf{R}_{23} | s'_2, s'_3 \rangle$ depend on the third set of spectral parameters belonging to the representation π_{α_3, β_3} involved in (2.104).

2.10.1. Heisenberg evolution operator

By construction, it is clear that Eq. (2.60) describes an *Hamiltonian evolution*, in the sense that there exists an invertible Heisenberg evolution operator \mathbf{U} ,

$$\mathcal{U}(\mathcal{O}) = \mathbf{U} \mathcal{O} \mathbf{U}^{-1} \quad (2.105)$$

which reduces the (discrete time) evolution to an internal automorphism of the algebra of observables. To write this operator explicitly, first remind that we have assumed that the elements z_k take only two values, as in (2.62). Next, consider the representation of algebra (2.56)

$$\pi_{\mathcal{O}} = \pi_{\alpha_1, \beta_1} \otimes \pi_{\alpha_2, \beta_2} \otimes \pi_{\alpha_1, \beta_1} \cdots \otimes \pi_{\alpha_2, \beta_2} \quad (2.106)$$

given by the product of the representations (2.88) with alternating values of the parameters α and β . Further, let

$$|\vec{s}\rangle = |s_1, \alpha_1, \beta_1\rangle \otimes |s_2, \alpha_2, \beta_2\rangle \otimes |s_3, \alpha_1, \beta_1\rangle \otimes \cdots \otimes |s_{2N}, \alpha_2, \beta_2\rangle \quad (2.107)$$

denotes the corresponding basis vectors. Then the matrix elements of \mathbf{U} are given by⁶

$$\langle \vec{s} | \mathbf{U} | \vec{s}' \rangle = \prod_{n=1}^N \langle s_{2n-1}, s_{2n} | \mathbf{R} | s'_{2n+1}, s'_{2n} \rangle. \quad (2.108)$$

The physical regime corresponds to imaginary values of α_i, β_i . In this case \mathbf{R} -matrix is unitary and therefore the Heisenberg evolution operator is unitary as well.

To write down the matrix elements of the evolution operator for $T \geq 1$ steps of the discrete time introduce $2N(T+1)$ variables $\{s_{k,t}\}$ with $k = 1, 2, \dots, 2N$ and $t = 0, 1, \dots, T$, such that

$$|\vec{s}\rangle_{(in)} = |\vec{s}, t\rangle|_{t=0}, \quad |\vec{s}\rangle_{(out)} = |\vec{s}, t\rangle|_{t=T}, \quad |\vec{s}, t\rangle = |s_{1,t}, s_{2,t}, \dots, s_{2N,t}\rangle \quad (2.109)$$

describes the initial and final states of the system. Then

$$\begin{aligned} Z &= {}_{(in)}\langle \vec{s} | \mathbf{U}^T | \vec{s} \rangle_{(out)} \\ &= \int \left\{ \prod_{t=0}^{T-1} \prod_{n=1}^N \langle s_{2n-1,t}, s_{2n,t} | \mathbf{R} | s_{2n+1,t+1}, s_{2n,t} \rangle \right\} \prod_{t=1}^{T-1} \left(\prod_{k=1}^{2N} ds_{k,t} \right). \end{aligned} \quad (2.110)$$

3. Quasi-classical limit and classical Yang–Baxter map

3.1. Poisson algebra

Consider the quasi-classical limit

$$q = e^{i\pi b^2} \rightarrow 1, \quad b \rightarrow 0, \quad (3.1)$$

where the parameter b is the same as in (2.82). In this limit the algebra (2.2) becomes a Poisson algebra (i.e., a commutative algebra with an associative Poisson bracket $\{, \}$, obeying the Leibniz rule). Making a substitution

$$K \rightarrow k, \quad E \rightarrow e, \quad F \rightarrow f, \quad (3.2)$$

⁶ The matrix elements of \mathbf{R} appearing in this formula are given by (2.94) with an appropriate substitution of the variables s_i and s'_i . The parameters $\alpha_{1,2}, \beta_{1,2}$ therein remain unchanged.

and replacing commutators in (2.2) by Poisson brackets,

$$[\ , \] \rightarrow 2\pi i b^2 \{ \ , \ \}, \quad b \rightarrow 0, \quad (3.3)$$

one obtains the following Poisson algebra,

$$\mathcal{P}: \quad \{k, e\} = ke, \quad \{k, f\} = -kf, \quad \{e, f\} = k - k^{-1}. \quad (3.4)$$

The central element (2.3) takes the form

$$C \rightarrow c = ef + k + k^{-1}. \quad (3.5)$$

The structure of the Hopf algebra in this limit remains intact. In particular, one can define the co-multiplication, co-unit and antipode, which, obey the same properties as for the quantum algebra $U_q(sl(2))$. We will denote them by the symbols $\overline{\Delta}$, $\overline{\epsilon}$ and \overline{S} , respectively. Define the co-multiplication $\overline{\Delta}$ as a map from the Poisson algebra \mathcal{P} to its tensor square

$$\overline{\Delta}: \quad \mathcal{P} \mapsto \mathcal{P} \otimes \mathcal{P}, \quad (3.6)$$

acting as

$$\overline{\Delta}(k) = k \otimes k, \quad \overline{\Delta}(e) = e \otimes k + 1 \otimes e, \quad \overline{\Delta}(f) = f \otimes 1 + k^{-1} \otimes f. \quad (3.7)$$

Similarly to (2.6) and (2.7) define the co-unit $\overline{\epsilon}$,

$$\overline{\epsilon}(k) = 1, \quad \overline{\epsilon}(e) = \overline{\epsilon}(f) = 0, \quad (3.8)$$

and the antipode \overline{S} ,

$$\overline{S}(k) = k^{-1}, \quad \overline{S}(e) = -ek^{-1}, \quad \overline{S}(f) = -kf. \quad (3.9)$$

Note that co-multiplication and co-unit are homomorphisms of the Poisson algebra, while the antipode is an antihomomorphisms. In particular, this means that

$$\overline{\Delta}(\{a, b\}) = \{\overline{\Delta}(a), \overline{\Delta}(b)\}, \quad \overline{S}(\{a, b\}) = -\{\overline{S}(a), \overline{S}(b)\}. \quad (3.10)$$

The relations (2.9) remain exactly the same as in the quantum case,

$$\overline{S}(1) = 1, \quad \overline{\epsilon} \circ \overline{S} = \overline{\epsilon}, \quad (\overline{S} \otimes \overline{S}) \circ \overline{\Delta} = \overline{\Delta}' \circ S, \quad (3.11)$$

where $\overline{\Delta}' = \sigma \circ \overline{\Delta}$ is another co-multiplication obtained from (3.7) by interchanging factors in the tensor product.

3.2. Quasi-classical expansion of the universal \mathbf{R} -matrix

In the limit (3.1) the universal \mathbf{R} -matrix (2.14) becomes singular. It is not difficult to show that

$$\mathbf{R} = (1 - e \otimes f)^{-\frac{1}{2}} \exp \left[\frac{1}{i\pi b^2} (\log k \otimes \log k + \text{Li}_2(e \otimes f)) \right] \left(1 + O(b^2) \right), \quad (3.12)$$

where k, e, f are the generators of the Poisson algebra (3.4) and $\text{Li}_2(x)$ is the Euler dilogarithm

$$\text{Li}_2(x) = - \int_0^x \frac{\log(1-t)}{t} dt. \quad (3.13)$$

3.3. Classical Yang–Baxter map

Even though the quasi-classical limit of the universal **R**-matrix becomes singular (see (3.12) above) its adjoint action on the elements of the tensor product of two algebras (2.2), appearing in (2.16), is well defined [17]. Therefore the $q \rightarrow 1$ limit of the quantum Yang–Baxter map (2.16) is well defined. We will denote it by a special symbol

$$\overline{\mathcal{R}} = \lim_{q \rightarrow 1} \mathcal{R}, \quad (3.14)$$

to distinguish it from its quantum counterpart.

Let $x = (k, e, f)$ denotes the set of generators of Poisson algebra \mathcal{P} and $x_{1,2}$ denotes the corresponding sets in the two components of the tensor product $\mathcal{P} \otimes \mathcal{P}$,

$$x_1 = x \otimes 1, \quad x_2 = 1 \otimes x, \quad x = \{k, e, f\}. \quad (3.15)$$

Similarly to (2.22) the classical map $\overline{\mathcal{R}}$ is completely defined by its action to two sets of generators in the tensor square $\mathcal{P} \otimes \mathcal{P}$,

$$\overline{\mathcal{R}}: \quad (x_1, x_2) \rightarrow (x'_1, x'_2), \quad (x'_1, x'_2) = \overline{\mathcal{R}}(x_1, x_2). \quad (3.16)$$

The formulae (2.17) lead to

$$\begin{cases} k'_1 = k_1 (1 - e_1 f_2 k_2 / k_1), \\ e'_1 = e_1 k_2, \\ f'_1 = f_1 / k_2 + f_2 - f_2 k_1^{-2} (1 - e_1 f_2 k_2 / k_1)^{-1}, \end{cases} \quad (3.17a)$$

and

$$\begin{cases} k'_2 = k_2 (1 - e_1 f_2 k_2 / k_1)^{-1}, \\ e'_2 = k_1 e_2 + e_1 - e_1 k_2^2 (1 - e_1 f_2 k_2 / k_1)^{-1}, \\ f'_2 = f_2 / k_1. \end{cases} \quad (3.17b)$$

We would like to stress that this map preserves the tensor product structure of two Poisson algebras

$$\overline{\mathcal{R}}: \quad \mathcal{P} \otimes \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{P}, \quad (3.18)$$

which means that the “primed” quantities $\{k'_i, e'_i, f'_i\}$, $i = 1, 2$, have exactly the same Poisson brackets (3.4) as the elements $\{k_i, e_i, f_i\}$. The inverse map reads

$$\begin{cases} k_1 = k'_1 (1 - e'_1 f'_2)^{-1}, \\ e_1 = e'_1 (k'_2)^{-1} (1 - e'_1 f'_2)^{-1}, \\ f_1 = (f'_1 + f'_2 / k'_1) (1 - e'_1 f'_2) k'_2 - k'_1 f'_2 k'_2, \end{cases} \quad (3.19a)$$

and

$$\begin{cases} k_2 = k'_2 (1 - e'_1 f'_2), \\ e_2 = (e'_2 + e'_1 k'_2) (1 - e'_1 f'_2) / k'_1 - e'_1 / (k'_1 k'_2) \\ f_2 = f'_2 k'_1 (1 - e'_1 f'_2)^{-1}. \end{cases} \quad (3.19b)$$

Similarly to the quantum case (see (2.32)) the direct and inverse maps are related by the antipode (3.9)

$$\overline{\mathcal{R}} \circ (\overline{S} \otimes \overline{S}) = (\overline{S} \otimes \overline{S}) \circ \overline{\mathcal{R}}^{-1}. \quad (3.20)$$

3.4. Properties of the classical Yang–Baxter map

The quasi-triangular structure of the quantum algebra is also inherited on the classical level. Therefore, most of the relations of Sect. 2.4 remains literally unchanged in the classical case. For completeness we summarize them below. Let

$$x_i = \{k_i, e_i, f_i\}, \quad i = 1, 2, 3, \dots \quad (3.21)$$

be sets of generating elements of different copies of the Poisson algebra (3.4). Introduce the set-theoretic multiplication $\overline{\delta}$ which acts on a two sets generators

$$\overline{\delta}: (x_1, x_2) \mapsto x', \quad (3.22)$$

and write it as

$$x' = \overline{\delta}(x_1, x_2). \quad (3.23)$$

Explicitly, one has

$$k' = k_1 k_2, \quad e' = e_1 k_2 + e_2, \quad f' = f_1 + k_1^{-1} f_2, \quad (3.24)$$

which is essentially the same formulae as (3.7). As in quantum case, the map (3.22) is only defined on two sets of generating element and the direction of the arrow in (3.22) is reversed with respect to (3.6).

Consider the tensor product $\mathcal{P} \otimes \mathcal{P} \otimes \mathcal{P}$ of three Poisson algebras (3.4) and let (x_1, x_2, x_3) denote three sets of generators in the corresponding components of the product. Define a functional operator $\overline{\mathcal{R}}_{12}$

$$\overline{\mathcal{R}}_{12}(x_1, x_2, x_3) := (\overline{\mathcal{R}}(x_1, x_2), x_3), \quad (3.25a)$$

which acts as (3.16) on the first two sets and does not affect the third one. Similarly, define operators

$$\overline{\mathcal{R}}_{13} = \sigma_{23} \circ \overline{\mathcal{R}}_{12} \circ \sigma_{23}, \quad \overline{\mathcal{R}}_{23} = \sigma_{12} \circ \overline{\mathcal{R}}_{13} \circ \sigma_{12}. \quad (3.25b)$$

In the $q \rightarrow 1$ limit Eq. (2.26) simply reduces to the set-theoretic Yang–Baxter equation for the classical maps $\overline{\mathcal{R}}_{ij}$

$$\overline{\mathcal{R}}_{23} \circ \overline{\mathcal{R}}_{13} \circ \overline{\mathcal{R}}_{12} = \overline{\mathcal{R}}_{12} \circ \overline{\mathcal{R}}_{13} \circ \overline{\mathcal{R}}_{23}. \quad (3.26)$$

To formulate the remaining properties of the classical map $\overline{\mathcal{R}}$ introduce additional notations,

$$\overline{\delta}_{12}(x_1, x_2, x_3) := (\overline{\delta}(x_1, x_2), x_3), \quad \overline{\delta}_{23}(x_1, x_2, x_3) := (x_1, \overline{\delta}(x_2, x_3)). \quad (3.27)$$

Then Eqs. (2.11) lead to

$$\bar{\delta}(x_2, x_1) = \bar{\delta} \circ \bar{\mathcal{R}}(x_1, x_2), \quad (3.28)$$

$$\bar{\mathcal{R}}(\bar{\delta}_{12}(x_1, x_2, x_3)) = \bar{\delta}_{12}(\bar{\mathcal{R}}_{23}(\bar{\mathcal{R}}_{13}(x_1, x_2, x_3))), \quad (3.29)$$

$$\bar{\mathcal{R}}(\bar{\delta}_{23}(x_1, x_2, x_3)) = \bar{\delta}_{23}(\bar{\mathcal{R}}_{12}(\bar{\mathcal{R}}_{13}(x_1, x_2, x_3))), \quad (3.30)$$

where the permutation operator σ is defined in (2.10). Further, from (2.13) it follows that

$$\bar{\mathcal{R}} \circ (\bar{\epsilon} \otimes 1) = (\bar{\epsilon} \otimes 1),$$

$$\bar{\mathcal{R}} \circ (1 \otimes \bar{\epsilon}) = (1 \otimes \bar{\epsilon}), \quad (3.31)$$

$$\bar{\mathcal{R}} \circ (\bar{S} \otimes \bar{S}) = (\bar{S} \otimes \bar{S}) \circ \bar{\mathcal{R}}^{-1}, \quad (3.32)$$

where the co-unit $\bar{\epsilon}$ and the antipode \bar{S} are defined in (3.9). We would like to stress that the above relations (3.26), (3.28)–(3.32) are not specific to the Poisson algebra (3.4) and must hold for any (quadratic) Poisson algebra which arise in the quasi-classical limit of a quasi-triangular Hopf algebra. In the case under consideration these relations could be verified using the explicit form of the map \mathcal{R} and its inverse given in (2.17) and (2.18). Note that, contrary to the quantum case, the above relations only involve ordinary substitutions for a set of commuting variables, since the Poisson algebra \mathcal{P} is a commutative algebra. The fact, that the map (3.18) is a “canonical transformation” (in the sense of Hamiltonian dynamics), preserving the Poisson brackets of the generating elements, does not play any role for the validity of (3.26)–(3.32).

3.5. r -matrix form of the Poisson algebra

As noted before, the universal \mathbf{R} -matrix (3.12) as well as the Yang–Baxter equations (2.12) and (2.34) becomes singular in the quasi-classical limit (3.1), (3.2). We will return to these singular equations in more details in Sect. 3.10.

Here we consider the remaining equations in Sect. 2.5, which all have a smooth limit $q \rightarrow 1$. The classical analog of the \mathbf{L} -operators (2.37) and (2.40) reads

$$\ell^+ = \begin{pmatrix} k^{\frac{1}{2}} & k^{\frac{1}{2}} f \\ 0 & k^{-\frac{1}{2}} \end{pmatrix}, \quad \ell^- = \begin{pmatrix} k^{-\frac{1}{2}} & 0 \\ -k^{-\frac{1}{2}} e & k^{\frac{1}{2}} \end{pmatrix}, \quad (3.33)$$

and

$$\ell(\lambda) = \lambda \ell^+ - \lambda^{-1} \ell^-. \quad (3.34)$$

For the R -matrix (2.39) one obtains

$$R^\pm = 1 \pm \frac{i\pi b^2}{2} + 2i\pi b^2 r^\pm + O(b^4), \quad b \rightarrow 0, \quad (3.35)$$

where

$$r^+ = \begin{pmatrix} 0 & \vdots & \\ & -\frac{1}{2} & 1 \\ - & - & - \\ & -\frac{1}{2} & \\ & \vdots & \\ & & 0 \end{pmatrix}, \quad r^- = \begin{pmatrix} 0 & \vdots & \\ & \frac{1}{2} & - \\ - & - & - \\ & -1 & \frac{1}{2} \\ & \vdots & \\ & & 0 \end{pmatrix}. \quad (3.36)$$

Similarly to (2.41) and (2.43) define

$$r(\lambda) = \lambda r^+ - \lambda^{-1} r^-, \quad \tilde{r}(\lambda) = r(\lambda) P \quad (3.37)$$

where P is the permutation operator and $r(\lambda)$ is the classical r -matrix satisfying the classical Yang–Baxter equation [43]

$$[r_{12}(\lambda), r_{13}(\lambda\mu)] + [r_{12}(\lambda), r_{23}(\mu)] + [r_{13}(\lambda\mu), r_{23}(\mu)] = 0. \quad (3.38)$$

The equation (2.45) then reduces to

$$\{\ell(\lambda) \otimes \ell(\mu)\} = [r(\lambda/\mu), \ell(\lambda) \otimes \ell(\mu)], \quad (3.39)$$

where the symbol $\{\otimes\}$ denotes Poisson bracket of two factor of the tensor product.

3.6. Heisenberg–Weyl realization (canonical variables)

The three-dimensional Poisson algebra (3.4) has one central element (3.5). Therefore it must essentially reduce to the classical Heisenberg–Weyl algebra, defined by the bracket

$$W : \quad \{u, v\} = uv. \quad (3.40)$$

The required coordinate transformation (cf. (2.48)) is

$$k = u, \quad e = v(z - u), \quad f = v^{-1}(1 - z^{-1}u^{-1}). \quad (3.41)$$

The variable z parametrizes the central element (3.5),

$$c = z + z^{-1}, \quad (3.42)$$

and has vanishing Poisson brackets with u and v . In the new variables the map (3.17) reduces to the classical analog of (2.51)

$$\begin{cases} u'_1 = u_1 g_{cl}, & u'_2 = u_2 / g_{cl}, \\ v'_1 = v_1 v_2 u_2 (v_1 u_2 + (v_2 - v_1 / z_2))^{-1}, & v'_2 = z_1 v_1 / z_2 + (v_2 - v_1 / z_2) u_1, \\ z'_1 = z_1, & z'_2 = z_2, \end{cases} \quad (3.43)$$

where

$$g_{cl} = 1 - \frac{v_1 (z_1 - u_1)(u_2 - 1/z_2)}{u_1 v_2}. \quad (3.44)$$

Note that the parameters $z_{1,2}$ remain unchanged under the map. Similarly for the inverse map one obtains

$$\begin{cases} u_1 = u'_1/g'_{cl}, & u_2 = u'_2 g'_{cl}, \\ v_1 = v'_1 v'_2 (z_1 v'_1/z_2 + (v'_2 - z_1 v'_1) u'_2)^{-1}, & v_2 = v'_1 + (v'_2 - z_1 v'_1)/u'_1, \end{cases} \quad (3.45)$$

where

$$g'_{cl} = 1 - \frac{v'_1 (z_1 - u'_1)(u'_2 - 1/z_2)}{v'_2 u'_2}. \quad (3.46)$$

3.7. Discrete classical evolution system

The quantum evolution system considered in Section 2.7 admits a straightforward classical limit. Remind, that the classical Yang–Baxter map (3.16) acts on the tensor square of the Poisson algebra (3.4),

$$\bar{\mathcal{R}}: (\mathcal{P}_1 \otimes \mathcal{P}_2) \mapsto (\mathcal{P}'_1 \otimes \mathcal{P}'_2) = \bar{\mathcal{R}}(\mathcal{P}_1 \otimes \mathcal{P}_2). \quad (3.47)$$

Correspondingly, the algebra of observables (2.56) becomes a tensor power Poisson algebra (3.4),

$$\mathcal{O}_{cl} = \mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \cdots \otimes \mathcal{P}_{2N-1} \otimes \mathcal{P}_{2N}, \quad N \geq 1. \quad (3.48)$$

Similarly to the quantum case the factors of this product are assigned to edges of one horizontal saw, corresponding to a fixed moment of the discrete time, as shown in Fig. 4 (one just need to replace all \mathcal{A} 's there by \mathcal{P} 's to get the classical picture). The classical evolution map is defined as

$$\mathcal{U}_{cl} = \mathcal{S} \circ (\tilde{\mathcal{R}}_{12} \circ \tilde{\mathcal{R}}_{34} \circ \cdots \circ \tilde{\mathcal{R}}_{(2N-1), 2N}), \quad (3.49)$$

where $\tilde{\mathcal{R}}_{ij} = \sigma_{ij} \circ \bar{\mathcal{R}}_{ij}$ is the map (3.47), acting on the i -th and j -th factors of the product (3.48), followed by the permutation operator σ_{ij} . The operator \mathcal{S} cyclically shifts (to the right) the factors in the product (3.48), similarly to (2.59). The one-step time evolution is defined as

$$\mathcal{O}'_{cl} = \mathcal{U}_{cl}(\mathcal{O}_{cl}) = \mathcal{P}'_1 \otimes \mathcal{P}'_2 \otimes \cdots \otimes \mathcal{P}'_{2N-1} \otimes \mathcal{P}'_{2N}. \quad (3.50)$$

For the realization (3.41) each edge will be assigned a triple of dynamical variables $\{u_i, v_i, z_i\}$, $i = 1, 2, \dots, 2N$. As in the quantum case we assume the homogeneous arrangement (2.62). Then all z 's stay unchanged under the time evolution. The remaining variables possess the canonical Poisson brackets

$$\{\log u_i, \log v_j\} = \delta_{ij}, \quad i, j = 1, 2, \dots, 2N, \quad (3.51)$$

which are preserved by the time evolution.

Taking classical limit in (2.63) and introducing discrete time variable t , one immediately obtains the Hamiltonian equations of motion

$$\begin{cases} u_{2n+1,t+1} = u_{2n-1,t} g_{n,t}, \\ v_{2n+1,t+1} = \left(\frac{1}{v_{2n,t}} + \left(\frac{1}{v_{2n-1,t}} - \frac{1}{z_2 v_{2n,t}} \right) \frac{1}{u_{2n,t}} \right)^{-1}, \\ u_{2n,t+1} = u_{2n,t} / g_{n,t}, \\ v_{2n,t+1} = \frac{z_1}{z_2} v_{2n-1,t} + \left(v_{2n,t} - \frac{1}{z_2} v_{2n-1,t} \right) u_{2n-1,t}, \end{cases} \quad (3.52)$$

where

$$g_{n,t} = 1 - \frac{v_{2n-1,t}(z_1 - u_{2n-1,t})(u_{2n,t} - 1/z_2)}{u_{2n-1,t}v_{2n,t}}. \quad (3.53)$$

3.8. Zero curvature representation

The quantum zero curvature representation (2.67) admits a straightforward quasi-classical limit. Let ℓ_1^\pm and ℓ_2^\pm denote the classical ℓ -operators (3.33) with elements belonging respectively to the first and second Poisson algebra of the tensor product $\mathcal{P}_1 \otimes \mathcal{P}_2$. Next, let $\tilde{\ell}_1^\pm$ and $\tilde{\ell}_2^\pm$ are defined in a similar way, but with elements from the “transformed” algebras \mathcal{P}'_1 and \mathcal{P}'_2 , arising from the map (3.47). Then Eqs. (2.67) lead to the relations

$$\ell_1^+ \ell_2^+ = \tilde{\ell}_2^+ \tilde{\ell}_1^+, \quad \ell_1^- \ell_2^+ = \tilde{\ell}_2^+ \tilde{\ell}_1^-, \quad \ell_1^- \ell_2^- = \tilde{\ell}_2^- \tilde{\ell}_1^-, \quad (3.54)$$

which can be easily verified using explicit formulae for the classical map (3.17). Conversely, the equations (3.54) can be used as an alternative definition of the classical Yang–Baxter map independent of the notion of the universal \mathbf{R} -matrix for the quantum algebra. The geometric interpretation of the relations (3.54) as the zero curvature representation remains essentially the same as in the quantum case, see Fig. 3(b) and the relevant discussion at the end of Sect. 2.8. Finally, note that Eqs. (3.54) can be conveniently rewritten as

$$\ell_1(\lambda) \ell_2^+ = \tilde{\ell}_2^+ \tilde{\ell}_1(\lambda), \quad \ell_1^- \ell_2(\lambda) = \tilde{\ell}_2(\lambda) \tilde{\ell}_1^-, \quad (3.55)$$

where λ is arbitrary, similarly to (2.69).

3.9. Involutive integrals of motion

The following considerations essentially repeat those for the quantum case in Sect. 2.9. The phase space of the classical evolution system (3.50) possesses $4N$ degrees of freedom ($2N$ coordinates and $2N$ momenta). Let us show that the system has exactly $2N$ integrals of motion (IM), which are in involution to each other with respect to the Poisson bracket (3.4). The classical analog of the transfer matrix in the quantum case is the *trace of the monodromy matrix* for an auxiliary linear problem. Similarly to (2.70) introduce two such traces

$$t(\lambda) = \text{Tr} \left(\ell_1(\lambda) \ell_2^+ \ell_3(\lambda) \ell_2^+ \cdots \ell_{2N-1}(\lambda) \ell_{2N}^+ \right) = \text{Tr} \prod_{n=1}^N \left(\ell_{2n-1}(\lambda) \ell_{2n}^+ \right), \quad (3.56a)$$

and

$$\bar{t}(\lambda) = \text{Tr} \left(\ell_1^- \ell_2(\lambda) \ell_3^- \ell_4(\lambda) \cdots \ell_{2N-1}^- \ell_{2N}(\lambda) \right) = \text{Tr} \prod_{n=1}^N \left(\ell_{2n-1}^- \ell_{2n}(\lambda) \right). \quad (3.56b)$$

It follows from (3.39) that these quantities form an involutive family

$$\{t(\lambda), t(\lambda')\} = \{\bar{t}(\lambda), \bar{t}(\lambda')\} = \{t(\lambda), \bar{t}(\lambda')\} = 0, \quad (3.57)$$

for arbitrary values of λ and λ' .

Further, using the zero curvature representation (2.69) one can easily show that $t(\lambda)$ and $\bar{t}(\lambda)$ are, indeed, time-independent, i.e., they are unchanged under the discrete evolution map (3.49),

$$\mathbf{t}(\lambda) = \mathcal{U}_{cl}(\mathbf{t}(\lambda)), \quad \bar{\mathbf{t}}(\lambda) = \mathcal{U}_{cl}(\bar{\mathbf{t}}(\lambda)). \quad (3.58)$$

Similarly to (2.77) in the quantum case, the classical Integrals of Motion can be defined as coefficients of expansions of the polynomials $\mathbf{t}(\lambda)$ and $\bar{\mathbf{t}}(\lambda)$ in the variable λ^2 .

3.10. The Lagrangian equation of motion and the action

In view of the quasi-classical correspondence, considered in the section, the Lagrangian function and the action of the classical evolution system (3.52) is determined quasi-classical asymptotics of the universal \mathbf{R} -matrix (3.12). More specifically, consider the expression for its matrix elements given by Eq. (2.94). It involves the spectral parameters $\alpha_{1,2}$, $\beta_{1,2}$ and the spin variables $s_{1,2}$ and $s'_{1,2}$. For a consistent quasi-classical limit these variables should become infinite, provided that the quantities

$$a_i = -2\pi i b \alpha_i, \quad b_i = -2\pi i b \beta_i, \quad \sigma_i = 2\pi b s_i, \quad \sigma'_i = 2\pi b s'_i, \quad i = 1, 2, \quad b \rightarrow 0, \quad (3.59)$$

remain finite for $b \rightarrow 0$. Introduce new functions

$$\lambda_a(\sigma) = -\frac{\sigma^2}{2} - \text{Li}_2(-e^{-\sigma-a}), \quad \bar{\lambda}_a(\sigma) = \frac{\sigma^2}{2} + \text{Li}_2(e^{-\sigma-a}), \quad (3.60)$$

where the Euler dilogarithm $\text{Li}_2(x)$ defined in (3.13). It is useful to note that

$$\frac{\partial \lambda_a(\sigma)}{\partial \sigma} = -\log(e^\sigma + e^{-a}), \quad \frac{\partial \bar{\lambda}_a(\sigma)}{\partial \sigma} = \log(e^\sigma - e^{-a}). \quad (3.61)$$

With these notations one can show that

$$\langle s_1, s_2 | R | s'_1, s'_2 \rangle = \exp \left\{ \frac{i}{2\pi b^2} \mathcal{L}(\sigma_1, \sigma_2, \sigma'_1, \sigma'_2) + O(1) \right\} \quad \text{as } b \rightarrow 0, \quad (3.62)$$

where

$$\mathcal{L} = \lambda_{b_1-a_2}(\sigma_2 - \sigma_1) + \lambda_{a_1-b_2}(\sigma'_2 - \sigma'_1) + \bar{\lambda}_{a_1-a_2}(\sigma'_2 - \sigma_1) + \bar{\lambda}_{b_1-b_2}(\sigma_2 - \sigma'_1). \quad (3.63)$$

This formula easily follows from the asymptotics of the functions (2.95) for $b \rightarrow 0$,

$$\begin{aligned} V_{\frac{ia}{2\pi b}} \left(\frac{\sigma}{2\pi b} \right) &= \exp \left\{ \frac{i}{2\pi b^2} \lambda_a(\sigma) + O(1) \right\}, \\ \bar{V}_{\frac{ia}{2\pi b}} \left(\frac{\sigma}{2\pi b} \right) &= \exp \left\{ \frac{i}{2\pi b^2} \bar{\lambda}_a(\sigma) + O(1) \right\}. \end{aligned} \quad (3.64)$$

Eqs. (3.52) are the (first order) Hamiltonian equations of motion for our dynamical system. To obtain the corresponding (second order) Lagrangian equations of motion one may consider the quasi-classical asymptotics of the matrix elements of the evolution operator (2.110). Substituting (3.62) therein and calculating the integral by the saddle point method one gets

$$\log Z = \frac{i}{2\pi b^2} \mathcal{A}(\{\sigma\}^{(cl)}) + O(1), \quad (3.65)$$

where $\mathcal{A}(\{\sigma\}^{(cl)})$ is the classical action of the system

$$\mathcal{A}(\{\sigma\}) = \sum_{n=1}^N \sum_{t=0}^{T-1} \mathcal{L}(\sigma_{2n-1,t}, \sigma_{2n,t}, \sigma_{2n+1,t+1}, \sigma_{2n,t+1}) \quad (3.66)$$

calculated on the stationary spin configuration $\{\sigma\}^{(cl)}$ defined as

$$\left. \frac{\partial \mathcal{A}(\{\sigma\})}{\partial \sigma_{k,t}} \right|_{\{\sigma\}=\{\sigma\}^{(cl)}} = 0, \quad k = 1, \dots, 2N, \quad t = 1, \dots, T-1. \quad (3.67)$$

The boundary spins $\{\sigma_{k,0}\}_{k=1}^{2N}$ and $\{\sigma_{k,T}\}_{k=1}^{2N}$ are kept fixed.

The same result can be obtained by the standard method of Hamiltonian mechanics. Remind that the classical map (3.43) involves the set of six independent variables $u_{1,2}, v_{1,2}, z_{1,2}$. They are simply related to the variables in (3.59),

$$z_i = -e^{b_i - a_i}, \quad v_i = e^{\sigma_i - b_i}, \quad v'_i = e^{\sigma'_i - b_i}. \quad (3.68)$$

Next, the map (3.43) is a canonical transformation, preserving the Poisson bracket

$$\{\log u_i, \log v_j\} = \delta_{ij}, \quad i, j = 1, 2. \quad (3.69)$$

Its generation function \mathcal{L} is defined by the equations

$$d\mathcal{L} = \sum_{i=1,2} (\log u'_i d \log v'_i - \log u_i d \log v_i) \quad (3.70)$$

or

$$\frac{\partial \mathcal{L}}{\partial \log v_i} = -\log u_i, \quad \frac{\partial \mathcal{L}}{\partial \log v'_i} = \log u'_i, \quad i = 1, 2. \quad (3.71)$$

Rewrite (3.43) in the form

$$\begin{aligned} u_1 &= z_1 \frac{1 - z_2 v'_2 / z_1 v_1}{1 - z_2 v_2 / v_1}, & u_2 &= z_2^{-1} \frac{1 - z_2 v_2 / v_1}{1 - v_2 / v'_1}, \\ u'_1 &= z_1 \frac{1 - v'_2 / z_1 v'_1}{1 - v_2 / v'_1}, & u'_2 &= z_2^{-1} \frac{1 - z_2 v'_2 / z_1 v_1}{1 - v'_2 / z_1 v'_1}. \end{aligned} \quad (3.72)$$

Integrating (3.70) and using the variables (3.68), one obtains the same Lagrangian (3.63).

3.11. Discrete Liouville equations

It is instructive to write down the Lagrangian equations of motion (3.67) explicitly. The corresponding Hamiltonian equations (3.52) are written in terms of the variables $u_{k,t}, v_{k,t}$, $k = 1, \dots, 2N$, where the variables $v_{k,t}$ (playing the role of coordinates) are related to $\sigma_{k,t}$ in (3.66) as follows

$$v_{2n-1,t} = e^{-b_1 + \sigma_{2n-1,t}}, \quad v_{2n,t} = e^{-b_2 + \sigma_{2n,t}}, \quad n = 1, \dots, N, \quad t = 0, \dots, T. \quad (3.73)$$

Their arrangement around an elementary quadrilateral is shown in Fig. 5. Rewriting (3.43) in the form (3.72) one obtains

$$\begin{aligned} u_{2n-1,t} &= z_1 \frac{1 - \frac{z_1}{z_2} \frac{v_{2n,t+1}}{v_{2n-1,t}}}{1 - z_2 \frac{v_{2n,t}}{v_{2n-1,t}}}, & u_{2n,t} &= \frac{1}{z_2} \frac{1 - z_2 \frac{v_{2n,t}}{v_{2n-1,t}}}{1 - \frac{v_{2n,t+1}}{v_{2n,t}}}, \\ u_{2n+1,t+1} &= z_1 \frac{1 - \frac{1}{z_1} \frac{v_{2n,t+1}}{v_{2n,t}}}{1 - \frac{v_{2n+1,t+1}}{v_{2n,t}}}, & u_{2n,t+1} &= \frac{1}{z_2} \frac{1 - \frac{z_2}{z_1} \frac{v_{2n,t+1}}{v_{2n-1,t}}}{1 - \frac{1}{z_1} \frac{v_{2n,t+1}}{v_{2n+1,t+1}}}. \end{aligned} \quad (3.74)$$

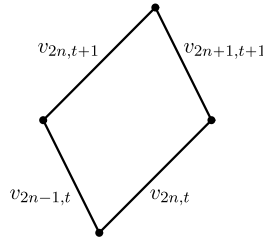


Fig. 5. Arrangement of classical variables $v_{2n-1,t}$ and $v_{2n,t}$.

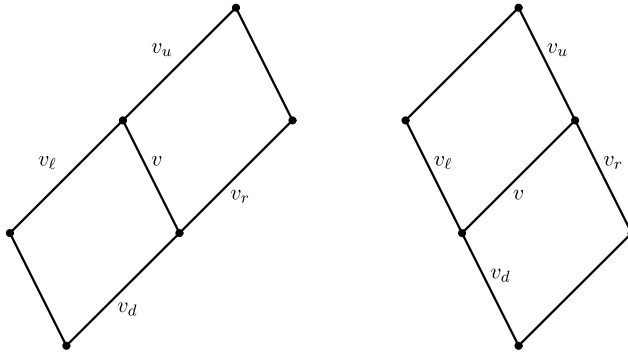


Fig. 6. Arrangement of variables for the Lagrangian equations of motion.

To obtain the Lagrangian equations of motion (3.67) one needs to exclude the variables $u_{k,l}$ (they are regarded as momentum variables). This is achieved by equating u 's defined by the first line of (3.74) to those from the second line of (3.74) (with an appropriate shift of indices). Consider a geometric structure of the resulting equations. Let $v = v_{k,t}$ be one of the v -variables for some fixed values of k, l . Define four adjacent variables

$$v_u = v_{k+1,t+1}, \quad v_d = v_{k-1,t-1}, \quad v_\ell = v_{k-1,t}, \quad v_r = v_{k+1,t}, \quad v = v_{k,t}, \quad (3.75)$$

shifted one site “up”, “down”, “left” or “right” with respect to v , as shown in Fig. 6. The equations (3.67) has different form depending on whether the spatial coordinate of the central site (k, t) is odd or even. For an odd $k = 2n - 1$ (left part of Fig. 6) one obtains

$$\left(1 - \frac{v_\ell}{z_1 v}\right) \left(1 - \frac{z_2 v_r}{v}\right) = \left(1 - \frac{z_2 v_u}{z_1 v}\right) \left(1 - \frac{v_d}{v}\right), \quad (3.76a)$$

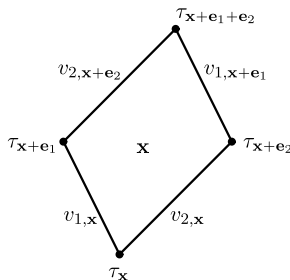
and for an even $k = 2n$ (right part of Fig. 6),

$$\left(1 - \frac{v}{z_1 v_r}\right) \left(1 - \frac{z_2 v}{v_\ell}\right) = \left(1 - \frac{z_2 v}{z_1 v_d}\right) \left(1 - \frac{v}{v_u}\right). \quad (3.76b)$$

3.11.1. General solution of the equation of motion

Below we present a general solution of both the Lagrangian and Hamiltonian equations of motion. It will be convenient to introduce the “light cone” coordinates:

$$\mathbf{x} = (n, t - n), \quad \mathbf{e}_1 = (1, 0), \quad \mathbf{e}_2 = (0, 1). \quad (3.77)$$

Fig. 7. Light cone coordinates, $\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$.

Conversely,

$$n = x_1, \quad t = x_1 + x_2, \quad \mathbf{x} = (x_1, x_2). \quad (3.78)$$

Moreover, we will use the variables $u_{1,2}(\mathbf{x})$ and $v_{1,2}(\mathbf{x})$ instead of $u_{k,t}$ and $v_{k,t}$,

$$u_1(\mathbf{x}) = u_{2n-1,t}, \quad u_2(\mathbf{x}) = u_{2n,t}, \quad v_1(\mathbf{x}) = v_{2n-1,t}, \quad v_2(\mathbf{x}) = v_{2n,t}. \quad (3.79)$$

The Hamiltonian equations of motion in these notations are easily obtained from (3.72) by substituting the variables $\{v_1, v_2, v'_1, v'_2\}$ therein with $\{v_1(\mathbf{x}), v_2(\mathbf{x}), v_1(\mathbf{x} + \mathbf{e}_1), v_2(\mathbf{x} + \mathbf{e}_1)\}$ and similarly for $\{u_1, u_2, u'_1, u'_2\}$. Their arrangement on lattice is shown in Fig. 7 (cf. Fig. 5). The resulting equations imply a simple relation

$$u_1(\mathbf{x})u_2(\mathbf{x}) = u_1(\mathbf{x} + \mathbf{e}_1)u_2(\mathbf{x} + \mathbf{e}_2), \quad (3.80)$$

which ensures the existence of the τ -function, defined by a system of two first order difference equation

$$u_1(\mathbf{x}) = \frac{\tau_{\mathbf{x}+\mathbf{e}_2}}{\tau_{\mathbf{x}}}, \quad u_2(\mathbf{x}) = \frac{\tau_{\mathbf{x}}}{\tau_{\mathbf{x}+\mathbf{e}_1}}. \quad (3.81)$$

With this substitution the system (3.74) can be rewritten as

$$\begin{aligned} \frac{v_1(\mathbf{x} + \mathbf{e}_1)}{v_1(\mathbf{x})} &= \frac{z_1 \tau_{\mathbf{x}} - \tau_{\mathbf{x}+\mathbf{e}_2}}{z_1 \tau_{\mathbf{x}+\mathbf{e}_1} - \tau_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2}}, \\ \frac{v_2(\mathbf{x} + \mathbf{e}_2)}{v_2(\mathbf{x})} &= \frac{z_2 \tau_{\mathbf{x}+\mathbf{e}_2} - \tau_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2}}{z_2 \tau_{\mathbf{x}} - \tau_{\mathbf{x}+\mathbf{e}_1}}, \\ \frac{v_1(\mathbf{x})}{v_2(\mathbf{x})} &= z_2 \frac{\tau_{\mathbf{x}+\mathbf{e}_1} \tau_{\mathbf{x}+\mathbf{e}_2} - \tau_{\mathbf{x}} \tau_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2}}{(z_1 \tau_{\mathbf{x}} - \tau_{\mathbf{x}+\mathbf{e}_2})(z_2 \tau_{\mathbf{x}} - \tau_{\mathbf{x}+\mathbf{e}_1})}. \end{aligned} \quad (3.82)$$

Here we have only three equations, since (3.80) is satisfied automatically by virtue of (3.81). Integrating the first two equations, one obtains

$$v_1(\mathbf{x}) = \frac{\beta_{x_2}}{z_1 \tau_{\mathbf{x}} - \tau_{\mathbf{x}+\mathbf{e}_2}}, \quad v_2(\mathbf{x}) = \frac{z_2 \tau_{\mathbf{x}} - \tau_{\mathbf{x}+\mathbf{e}_1}}{z_2 \alpha_{x_1}}, \quad (3.83)$$

where $\mathbf{x} = (x_1, x_2)$ are the light cone coordinates,

$$\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2, \quad x_1 = n, \quad x_2 = t - n, \quad (3.84)$$

and $\alpha_{x_1}, \beta_{x_2}$ are arbitrary constants of integration. The third equation in (3.82) reduces to a single equation for the τ -function,

$$\tau_{\mathbf{x}+\mathbf{e}_1} \tau_{\mathbf{x}+\mathbf{e}_2} - \tau_{\mathbf{x}} \tau_{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2} = \alpha_{x_1} \beta_{x_2}. \quad (3.85)$$

This is the inhomogeneous discrete Liouville equation, involving two arbitrary functions α_{x_1} and β_{x_2} . The corresponding geometric arrangement of the τ -functions is shown in Fig. 7. This is a generalization of the homogeneous discrete Liouville equation (with $\alpha_{x_1} \equiv 1$, $\beta_{x_2} \equiv 1$), previously studied in [19–22]. It appears that the inhomogeneity does not bring notable complications. Indeed, a general solution to Eq. (3.85) is given by the formula

$$\tau_{\mathbf{x}} = \frac{1 + f_{x_1} g_{x_2}}{\phi_{x_1} \gamma_{x_2}}, \quad (3.86)$$

where ϕ_{x_1} , γ_{x_2} are arbitrary functions, while f_{x_1} and g_{x_2} are determined by the following first order difference equations

$$f_{x_1+1} - f_{x_1} = \alpha_{x_1} \phi_{x_1} \phi_{x_1+1}, \quad g_{x_2+1} - g_{x_2} = \beta_{x_2} \gamma_{x_2} \gamma_{x_2+1}. \quad (3.87)$$

To summarize the formulae (3.81), (3.83), (3.86) and (3.87), together with the definitions (3.73) and (3.79), provide the most general solution to both the Lagrangian equations (3.67), (3.76) as well as the Hamiltonian equations of motion (3.74). The solution contains $4N$ arbitrary constants α_{x_1} , β_{x_2} , ϕ_{x_1} , γ_{x_2} ($x_1, x_2 = 1, \dots, N$), whose number precisely coincides with the dimension of the phase space of our classical discrete evolution system (3.50).

4. Conclusion

In this paper we have shown that the theory of integrable maps (and more specifically the so-called Yang–Baxter maps) can be naturally included as a part of the modern theory of integrable system, based on the theory of quantum groups, the Quantum Inverse Problem Method and their classical counterparts. The considerations apply to both quantum and classical theories (the latter arise as the quasi-classical limit of the quantum case). One of the main advantages of our algebraic approach is that the resulting discrete integrable evolution systems automatically possess meaningful Hamiltonian structures. In addition, the problem of quantization of the Yang–Baxter maps appears to be solved from the very beginning.

We have illustrated the entire scheme on the example on the quantized Lie algebra $U_q(sl(2))$ (and its Poisson algebra limit for the classical case). Completely parallel presentations of the quantum and classical cases are given in Sect. 2 and Sect. 3 respectively. In particular, a general solution the classical discrete Liouville equations arising in this context (both in the Lagrangian and Hamiltonian forms) is given in Sect. 3.11.

Although the above considerations were mainly restricted to the case of $U_q(sl(2))$, our approach is rather general and can be applied to any quantized Lie (super) algebra as well as their affine extensions. We hope to consider some of these problems in future publications.

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Appendix A. Additional properties of the R-matrix

Here we present some additional relations connected with the **R**-matrix (2.94). The non-compact quantum dilogarithm (2.96) (sometimes called the Barnes double-sine function) is a unique solution of the functional equation

$$\log \varphi(z - i\frac{b}{2}) - \log \varphi(z + i\frac{b}{2}) = \log(1 + e^{2\pi b z}) \quad (\text{A.1})$$

provided that $\varphi(z)$ is analytic in the strip

$$-\text{Re}(b) \leq \text{Im}(z) \leq \text{Re}(b). \quad (\text{A.2})$$

It satisfies the relations

$$\varphi(z)\varphi(-z) = e^{i\pi z^2 - i\pi(1-2\eta^2)/6}, \quad \varphi(z)|_{z \rightarrow -\infty} \rightarrow 1, \quad \varphi(z)^* = \frac{1}{\varphi(z^*)}. \quad (\text{A.3})$$

Define also another function

$$\Phi(z) = \exp \left(\frac{1}{8} \int_{\mathbb{R}+i0} \frac{e^{-2izw}}{\sinh(wb) \sinh(w/b) \cosh(2\eta w)} \frac{dw}{w} \right), \quad (\text{A.4})$$

which possesses the properties

$$\Phi(z)\Phi(-z) = e^{i\pi z^2/2 - i\pi(1-8\eta^2)/12}, \quad \Phi(z)|_{z \rightarrow -\infty} \rightarrow 1. \quad (\text{A.5})$$

The Boltzmann weights for the Faddeev–Volkov model (which is related to the quantized affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$) are defined as [38],

$$W_\alpha(s) = \frac{1}{F_\alpha} e^{2\pi\alpha s} \frac{\varphi(s + i\alpha)}{\varphi(s - i\alpha)}, \quad \overline{W}_\alpha(s) = W_{\eta-\alpha}(s), \quad (\text{A.6})$$

where

$$F_\alpha = e^{i\pi\alpha^2 + i\pi(1-8\eta^2)/24} \Phi(2i\alpha). \quad (\text{A.7})$$

It is useful to note that $W_\alpha(s) = W_\alpha(-s)$. Consider also a simple limit,

$$s \rightarrow s - K, \quad i\alpha \rightarrow i\alpha - K, \quad K \rightarrow \infty. \quad (\text{A.8})$$

In this case, from (A.3) and (A.5) it follows that

$$W_\alpha(s) \rightarrow e^{-i\pi K^2 + 2\pi i K s} V_\alpha(s), \quad \overline{W}_\alpha(s) \rightarrow e^{i\pi K^2 - 2\pi i K s} \overline{V}_\alpha(s), \quad (\text{A.9})$$

where $V_\alpha(s)$ and $\overline{V}_\alpha(s)$ are defined in (2.95).

The weights $W_\alpha(s)$ and $\overline{W}_\alpha(s)$ satisfy the following star–triangle relation [38,42],

$$\int_{\mathbb{R}} d\sigma \overline{W}_\alpha(a - \sigma) W_{\alpha+\beta}(c - \sigma) \overline{W}_\beta(b - \sigma) = W_\beta(a - c) \overline{W}_{\alpha+\beta}(a - b) W_\alpha(c - b) \quad (\text{A.10})$$

which admits several interesting limits. For example, if

$$a \rightarrow a - K, \quad c \rightarrow c - K, \quad i\alpha \rightarrow i\alpha - K, \quad K \rightarrow \infty, \quad (\text{A.11})$$

then the singular terms in (A.10) cancel out and the relation reduces to

$$\int_{\mathbb{R}} d\sigma \overline{V}_\alpha(a - \sigma) V_{\alpha+\beta}(c - \sigma) \overline{W}_\beta(b - \sigma) = W_\beta(a - c) \overline{V}_{\alpha+\beta}(a - b) V_\alpha(c - b). \quad (\text{A.12})$$

A similar limit,

$$a \rightarrow a + K, \quad c \rightarrow c + K, \quad i\alpha \rightarrow i\alpha - K, \quad K \rightarrow \infty, \quad (\text{A.13})$$

yields

$$\int_{\mathbb{R}} d\sigma \bar{V}_\alpha(\sigma - a) V_{\alpha+\beta}(\sigma - c) \bar{W}_\beta(b - \sigma) = W_\beta(a - c) \bar{V}_{\alpha+\beta}(b - a) V_\alpha(b - c). \quad (\text{A.14})$$

These relations can be recursively used to prove the Yang–Baxter equation (2.104) given in the main text.

It is interesting to study the classical limit of the above relations (A.12) and (A.14). The classical limits of functions V and \bar{V} is given by (3.64). Similar limits of W and \bar{W} are given by

$$\begin{aligned} W_{\frac{ia}{2\pi b}}\left(\frac{\sigma}{2\pi b}\right) &= \exp\left\{\frac{i}{2\pi b^2}\Lambda_a(\sigma) + O(1)\right\}, \\ \bar{W}_{\frac{ia}{2\pi b}}\left(\frac{\sigma}{2\pi b}\right) &= \exp\left\{\frac{i}{2\pi b^2}\bar{\Lambda}_a(\sigma) + O(1)\right\}, \end{aligned} \quad (\text{A.15})$$

where

$$\frac{\partial \Lambda}{\partial \sigma} = \log \frac{e^\sigma + e^\alpha}{1 + e^{\sigma+\alpha}}, \quad \frac{\partial \bar{\Lambda}}{\partial \sigma} = \log \frac{e^\sigma - e^{-\alpha}}{1 - e^{\sigma-\alpha}}. \quad (\text{A.16})$$

Then, for instance, the stationary point of the integral (A.14) in the limit $b \rightarrow 0$ is determined by the relation

$$\frac{\partial}{\partial \sigma} (\bar{\lambda}_\alpha(\sigma - a) + \lambda_{\alpha+\beta}(\sigma - c) + \bar{\lambda}_\beta(\sigma - b)) = 0, \quad (\text{A.17})$$

which is equivalent to

$$\log(e^{\sigma-a} - e^{-\alpha}) - \log(e^{\sigma-c} + e^{-\alpha-\beta}) + \log \frac{e^{\sigma-b} - e^{-\beta}}{1 - e^{\sigma-b-\beta}} = 0. \quad (\text{A.18})$$

The last relation has the structure of the Adler–Bobenko–Suris [12] tree-legs equation

$$\psi(x_0, y_0; \alpha_0) - \psi(x_0, y_1; \alpha_1) = \varphi(x_0, x_1; \alpha_0, \alpha_1) \quad (\text{A.19})$$

corresponding to the H_3 system in their classification.

Appendix B. Relation to R -matrix for quantum Teichmüller theory

The \mathbf{R} -matrix (2.99) coincides with the inverse of \mathbf{R} -matrix (44) from [39]. Details are the following. Moving quadratic exponents in (2.99) to the right, one could obtain the left hand side of the following identity:

$$ie^{-i\pi(\mathbf{x}_1 - \mathbf{x}_2)^2} e^{-i\pi(\mathbf{p}_1^2 + \mathbf{p}_2^2)} e^{-i\pi(\mathbf{x}_1 - \mathbf{x}_2)^2} \mathbf{P}_{12} = e^{-2\pi i \mathbf{p}_1 \mathbf{p}_2}. \quad (\text{B.1})$$

This identity can be easily proven by consideration of a kernel or of a normal symbol of the left and right hand sides and performing the Gauss integration.

Using the Pentagon identity (2.97) twice in (2.99), one can then bring \mathbf{R}_{12} to the form

$$\mathbf{R}_{12} = \mathbf{F}_{12} \varphi(i(\beta_1 - \alpha_2) + \mathbf{x}_1 - \mathbf{x}_2) \mathbf{F}_{12}^{-1} e^{-2\pi i \mathbf{p}_1 \mathbf{p}_2}, \quad (\text{B.2})$$

where

$$\mathbf{F}_{12} = \varphi(i(\alpha_1 - \beta_1) + \mathbf{p}_1) \varphi(i(\alpha_2 - \beta_2) - \mathbf{p}_2). \quad (\text{B.3})$$

This form of \mathbf{R}_{12} literally coincides with \mathbf{R}^{-1} , formula (44) from [39], after indentifying corresponding variables. The authors are grateful to R.M. Kashaev for this observation.

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